CS5319 Advanced Discrete Structure

Homework 1

Due: 1:10 pm, October 7, 2010 (before class)

1. From n distinct integers, two groups of integers are to be selected with x integers in the first group and y integers in the second group, where $x + y \leq n$. In how many ways can the selection be made such that the smallest integer in the first group is larger than the largest integer in the second group?

2. In how many ways can four distinct integers be selected from 1 to 300 such that their sum is a multiple of 3?

3. There are 15 lines on the plane.

   (a) Suppose that four lines are parallel to each other, while any two of the others have an intersection. How many line segments are there?

   (b) Suppose that four lines are parallel to each other, five other lines meet exactly at one point. For other pairs of lines, they have a distinct intersection. How many line segments are there?

4. Suppose that no three of the diagonals of a convex n-gon meet at the same point inside of the n-gon. Triangles will be formed with the sides of made up of the sides of the n-gon, the diagonals, or segments of the diagonals. How many different triangles are there?

   For instance, when $n = 4$, there will be 8 triangles.

5. Consider a city whose roads and junctions look like an integral grid, with lower-left corner $(0, 0)$ and upper-right corner $(2n, 2r)$. We want to travel from the lower-left corner to the upper-right corner, with each step moving either 1 unit right or 1 unit up. However, the city center (n, r) is now having construction, so that we are forbidden to step on it. In how many ways can we perform the above traveling?

6. A ternary string is a string with characters drawn from the alphabet $\{0, 1, 2\}$.

   (a) Show that the number of length-n ternary strings with even number of 0 is $(3^n + 1)/2$.

   (b) Prove the following identity using combinatorial arguments:

   $$\binom{n}{0}2^n + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{q}2^{n-q} = \frac{3^n + 1}{2},$$

   where $q = n$ if n is even, and $q = n - 1$ if n is odd.

7. Suppose we have n pairs of weights, each pair having two equivalent weights. We can use them for weighing items. For example, if we have two pairs of weights $(1, 1)$ and $(2, 2)$, then we can measure items with integral weights ranging from 1 to 6.

   Suppose now we need to measure items with integral weights ranging from 1 to 100.

   (a) Show that least 5 pairs of weights are needed.

   (b) Find 5 pairs of weights that can serve our purpose.

8. (Challenging: No marks) How many permutations of the integers $1, 2, \ldots, n$ are there such that every integer is followed by (but not necessarily immediately followed by) an integer which differs from it by 1?

   For example, with $n = 4$, 1432 is an acceptable permutation but 2431 is not.