CS 5319
Advanced Discrete Structure

Lecture 6:
Recurrence Relations II

Outline

- Introduction
- Linear Recurrence Relations with Constant Coefficients
- Solution by Generating Functions
- ★ Special Non-Linear Recurrences
- Recurrence with Two Indices

This Lecture

Special Non-Linear Recurrences

Special Non-Linear Recurrences

- So far, we have only discussed how to solve linear recurrence with constant coefficients
- When recurrence is non-linear, or with variable coefficients, it becomes very hard to solve
- One exception is the following common type of recurrence, which can be solved by GF :

$$a_n = a_{n-r}b_0 + a_{n-r-1}b_1 + \ldots + a_0 b_{n-r}$$

Special Non-Linear Recurrences

- Consider the simplest example of this class :

$$a_n = a_{n-r}a_0 + a_{n-r-1}a_1 + \ldots + a_0 a_{n-r}$$

- Suppose the recurrence is valid for all $n \geq k$
- If we multiply both sides by x^n, and summing from $n = k$ to $n = \infty$, we obtain

$$\sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} (a_{n-r}a_0 + a_{n-r-1}a_1 + \ldots + a_0 a_{n-r})\, x^n$$

Special Non-Linear Recurrences

- Observe that in the previous sum, the coefficient of x^n is exactly the coefficient of x^r in $A(x)A(x)$
  - Here, $A(x)$ = GF for $(a_0, a_1, a_2, \ldots)$
- Thus we have :

$$\text{LHS} = A(x) - a_0 - a_1 x - \ldots - a_{k-1} x^{k-1}$$

$$\text{RHS} = x^r \Big[\, A(x)\, A(x) - a_0^{\,2} - (a_1 a_0 + a_0 a_1)\, x - \ldots - (a_{k-r-1} a_0 + a_{k-r-1} a_1 + \ldots + a_0 a_{k-r-1})\, x^{k-r-1} \,\Big]$$

Special Non-Linear Recurrences

- The previous observation thus allows us to obtain a quadratic equation in $A(x)$, so that we can then solve $A(x)$ by ordinary algebra
- Note that the boundary conditions consists of the values of $a_0, a_1, a_2, \ldots, a_{k-1}$

Special Non-Linear Recurrences

- Ex : Find the # of ways to parenthesize the expression $w_1 + w_2 + \ldots + w_n$
- For instance, for $n = 4$, there are 5 ways :

1. $(w_1 + w_2) + (w_3 + w_4)$
2. $w_1 + (w_2 + (w_3 + w_4))$
3. $w_1 + ((w_2 + w_3) + w_4)$
4. $(w_1 + (w_2 + w_3)) + w_4$
5. $((w_1 + w_2) + w_3) + w_4$

Special Non-Linear Recurrences

- Let a_n = # ways to parenthesize $w_1 + \ldots + w_n$
- It follows that for $n > 1$ (with $a_1 = 1$)

$$a_n = a_{n-1}a_1 + a_{n-2}a_2 + \ldots + a_1 a_{n-1}$$

- Since we can choose a_0 arbitrarily without affecting the result, we set $a_0 = 0$ so that for $n > 1$

$$a_n = a_n a_0 + a_{n-1}a_1 + \ldots + a_0 a_n$$

Special Non-Linear Recurrences

- Thus we have :

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (a_n a_0 + a_{n-1} a_1 + \ldots + a_0 a_n)\, x^n$$

- Consequently, we have :

$$\text{LHS} = A(x) - a_1 x - a_0$$

$$\text{RHS} = A(x)\, A(x) - a_0^2 - (a_1 a_0 + a_0 a_1)\, x$$

Special Non-Linear Recurrences

- By re-arranging terms, we get :

$$[A(x)]^2 - A(x) + x = 0$$

- Solving for $A(x)$, we obtain

$$A(x) = \frac{1 \pm \sqrt{1 - 4x}}{2}$$

Special Non-Linear Recurrences

- Although there are two solutions for $A(x)$, only one will give a +ve sequence of coefficients
- Precisely, the general term of $(1 - 4x)^{1/2}$ is :

$$C(1/2\ ,\ n)\ (-\ 4x)^n = -\ (2\ /\ n)\ C(2n-2,\ n-1)\ x^n$$

- This implies $A(x) = (\ 1 - (1 - 4x)^{1/2}\ )\ /\ 2$, and

$$a_n = C(2n-2,\ n-1)\ /\ n \qquad \text{for } n > 0$$

Special Non-Linear Recurrences

- Next we consider the recurrence of the form :

$$b_n = a_{n-r}b_0 + a_{n-r-1}b_1 + \ldots + a_0 b_{n-r}$$

- Suppose the recurrence is valid for all $n \geq k$
- If we multiply both sides by x^n, and summing from $n = k$ to $n = \infty$, we obtain

$$\sum_{n=k}^{\infty} b_n x^n = \sum_{n=k}^{\infty} (a_{n-r}b_0 + a_{n-r-1}b_1 + \ldots + a_0 b_{n-r})\, x^n$$

Special Non-Linear Recurrences

- Thus we have :

$$\text{LHS} = B(x) - b_0 - b_1 x - \ldots - b_{k-1} x^{k-1}$$

$$\text{RHS} = x^r \left[A(x)B(x) - a_0 b_0 - (a_1 b_0 + a_0 b_1)\, x - \ldots - (a_{k-r-1} b_0 + a_{k-r-2} b_1 + \ldots + a_0 b_{k-r-1})\, x^{k-r-1} \right]$$

where $B(x)$ is the GF for $(b_0, b_1, b_2, \ldots)$

- If either $A(x)$ or $B(x)$ is known, the other can be computed

Special Non-Linear Recurrences

Ex: Pattern Occurrences in Binary String

- Suppose we get a binary string S and a pattern P
- Consider the following procedure :

1. Scan S to detect the first occurrence of P
2. If P is detected, say after the k th bit, the scanning start over at the $(k+1)$ th bit to detect the next occurrence of P
3. Repeat Step 2 until S is completely scanned

Special Non-Linear Recurrences

- E.g., suppose that

$$S = 1101010101011, \ \ P = 010$$

- Using the previous procedure, we detect two occurrences of P, which are located when 5th bit and when 9th bit of S are scanned
  - In contrast, the occurrences of 010 is not detected when 7th bit or 11th bit is scanned

Special Non-Linear Recurrences

Q: Find the # of n-bit binary strings such that 010 is detected when the nth bit is scanned

A: Let b_n = # of such sequences

For all strings ending with 010, they can be divided into two groups:

1. 010 is detected at nth bit (b_n of them)
2. 010 not detected at nth bit (how many?)

Special Non-Linear Recurrences

- It follows that for $n \geq 5$

$$2^{n-3} = b_{n-2} + b_n$$

- Since we can choose b_0, b_1, b_2 arbitrarily without affecting the result, we shall set

$$b_1 = b_2 = 0$$

  so that the above recurrence is valid for $n \geq 3$

  (And we set $b_0 = 1$ for convenience)

Special Non-Linear Recurrences

- Thus

$$\sum_{n=3}^{\infty} 2^{n-3} x^n = \sum_{n=3}^{\infty} (b_{n-2} + b_n)\, x^n$$

and

$$x^3 / (1 - 2x) = x^2(B(x) - 1) + (B(x) - 1)$$

- Consequently, we have :

$$B(x) = \frac{1 - 2x + x^2 - x^3}{1 - 2x + x^2 - 2x^3} = 1 + x^3 + 2x^4 + 3x^5 + 6x^6 + \ldots$$

Special Non-Linear Recurrences

Q: Find the # of n-bit binary strings such that 010 is *first* detected at the nth bit

A: Let a_n = # of such sequences

Let b_n = # of sequences with 010 detected at the nth bit

- Then we have for $n \geq 6$:

$$b_n = a_n + a_{n-3}b_3 + a_{n-4}b_4 + \ldots + a_3 b_{n-3}$$

Special Non-Linear Recurrences

- Since we can arbitrarily choose $a_0, a_1, a_2, b_0, b_1, b_2$ without affecting the result, we shall set

$$a_0 = a_1 = a_2 = 0, \quad b_0 = 1, b_1 = b_2 = 0$$

so that we can "simplify" the above recurrence as follows, and it will also be valid for $n \geq 3$:

$$b_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \ldots + a_0 b_n$$

Special Non-Linear Recurrences

- By multiplying both sides with x^n, and summing for all $n \geq 3$, we obtain :

$$B(x) - 1 = A(x)\, B(x)$$

- Since $B(x)$ was already known, we get :

$$A(x) = 1 - \frac{1}{B(x)} = \frac{x^3}{1 - 2x + x^2 - x^3}$$

Special Non-Linear Recurrences

- In general, for a pattern P of length p, suppose

$$A(x) = \text{GF for } P \text{ to } \textit{first} \text{ occur at } n\text{th bit}$$
$$B(x) = \text{GF for } P \text{ to occur at } n\text{th bit}$$

where $a_0 = a_1 = a_2 = \ldots = a_{p-1} = 0,$

$b_0 = 1, \quad b_1 = b_2 = \ldots = b_{p-1} = 0$

- Then we have :

$$b_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \ldots + a_0 b_n$$

Special Non-Linear Recurrences

- Consequently, by multiplying both sides with x^n, and summing for all $n \geq p$, we obtain :

$$B(x) - 1 = A(x)\, B(x)$$

so that if $B(x)$ is known, then we get :

$$A(x) = 1 - \frac{1}{B(x)}$$

Special Non-Linear Recurrences

Q: Find the # of n-bit binary strings such that an occurrence of 010 is followed by an occurrence of 110

Here, the procedure of scanning is as follows.

1. A string is scanned to detect the *first* occurrence of 010
2. If 010 is detected, say after the k th bit, the scanning start over at the $(k+1)$ th bit to detect the occurrence of 110

Special Non-Linear Recurrences

A: Let a_n = # of n-bit strings with 010 first occurs at nth bit

Let b_n = # of n-bit strings with 110 occurs

Let c_n = # of n-bit strings with 010 occurs and then 110 occurs

Then we have for $n \geq 6$:

$$c_n = a_{n-3}b_3 + a_{n-4}b_4 + \ldots + a_3 b_{n-3}$$

Special Non-Linear Recurrences

- Since we can arbitrarily choose $a_0, a_1, a_2, b_0, b_1, b_2$ without affecting the result, we shall set

$$a_0 = a_1 = a_2 = 0, \quad b_0 = b_1 = b_2 = 0$$

so that we can "simplify" the above recurrence as follows, and it will also be valid for $n \geq 6$:

$$c_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \ldots + a_0 b_n$$

Special Non-Linear Recurrences

- Further, we can set $c_0 = c_1 = \ldots = c_5 = 0$ without affecting the result
- Consequently, by multiplying both sides with x^n, and summing for all $n \geq 6$, we obtain :

$$C(x) = A(x)\,B(x)$$

- $A(x)$ was already computed in the previous example ➔ It remains to find $B(x)$

Special Non-Linear Recurrences

- To find $B(x)$, we define

  d_n = # of n-bit strings with 110 first occurs at nth bit

- Then we have for $n \geq 3$:

$$b_n = d_3 \times 2^{n-3} + d_4 \times 2^{n-4} + \ldots + d_{n-1} \times 2 + d_n$$

Special Non-Linear Recurrences

- Next we shall set

$$d_0 = d_1 = d_2 = 0$$

so that we can "simplify" the above recurrence as follows, and it will also be valid for $n \geq 3$:

$$b_n = d_0 \times 2^n + d_1 \times 2^{n-1} + \ldots + d_{n-1} \times 2 + d_n$$

Special Non-Linear Recurrences

- Consequently, by multiplying both sides with x^n, and summing for all $n \geq 3$, we obtain :

$$B(x) = D(x)\,(1 - 2x)^{-1}$$

- On the other hand, as we have previously shown, $D(x)$ can be computed by finding the GF $E(x)$ for 110 to occur, so that $D(x) = 1 - 1 / E(x)$

➔ we can obtain $D(x)$, thus $B(x)$, thus $C(x)$

Special Non-Linear Recurrences

- In particular, we have :

$$\begin{aligned} E(x) &= 1 + 2x^3 + 4x^4 + 8x^5 + \ldots \\ &= 1 + x^3\,(1 - 2x)^{-1} \end{aligned}$$

and

$$D(x) = 1 - \frac{1}{E(x)} = \frac{x^3}{1 - 2x + x^3}$$

Special Non-Linear Recurrences

- Therefore, we have :

$$C(x) = A(x)\, B(x) = A(x)\, D(x)\, (1 - 2x)^{-1}$$

$$= \frac{x^3}{1 - 2x + x^2 - x^3} \; \frac{x^3}{1 - 2x + x^3} \; \frac{1}{1 - 2x}$$

$$= \frac{x^6}{1 - 6x + 13x^2 - 12x^3 + 4x^4 + x^5 - 3x^6 + 2x^7}$$

$$= x^6 + 6x^7 + 23x^8 + 72x^9 + \ldots$$

Recurrence with Two Indices

Recurrence with Two Indices

- Recall the following identity :

$$C(n, r) = C(n-1, r) + C(n-1, r-1)$$

- This is a recurrence relation with two indices
- The boundary conditions include $C(n, 0) = 1$ for all $n \geq 0$, and $C(0, r) = 0$ for all $r > 0$
- Then for $n \geq 1$, $r \geq 1$, any value of $C(n, r)$ can be computed recursively

Recurrence with Two Indices

- To solve a recurrence with two indices by GF technique, we first define a GF for each value of one of the indices :

$$A_0(x) = a_{0,0} + a_{0,1}\,x + a_{0,2}\,x^2 + \ldots$$

$$A_1(x) = a_{1,0} + a_{1,1}\,x + a_{1,2}\,x^2 + \ldots$$

$$A_2(x) = a_{2,0} + a_{2,1}\,x + a_{2,2}\,x^2 + \ldots$$

$$\ldots$$

Recurrence with Two Indices

- Next, we define a GF $\mathcal{A}(y, x)$ for

$$(A_0(x), A_1(x), A_2(x), \ldots),$$

using powers of y as indicators :

$$\mathcal{A}(y, x) = A_0(x) + A_1(x)\,y + A_2(x)\,y^2 + \ldots$$

- Consequently, we have

$$a_{i,j} = \text{coefficient of } y^i x^j$$

Recurrence with Two Indices

Ex: Find the GF for $C(n, r)$

$$F_n(x) = C(n,0) + C(n,1)\,x + C(n,2)\,x^2 + \ldots$$

- From the previous recurrence of $C(n, r)$, we get:

$$\sum_{r=1}^{\infty} C(n, r)\,x^r = \sum_{r=1}^{\infty} \big(C(n-1, r) + C(n-1, r-1)\big)\,x^r$$

Recurrence with Two Indices

- Thus we have

$$F_n(x) - C(n, 0) = F_{n-1}(x) - C(n-1, 0) + x\, F_{n-1}(x)$$

- By simplifying terms, we get :

$$\begin{aligned} F_n(x) &= (1 + x)\, F_{n-1}(x) = \ldots \\ &= (1 + x)^n F_0(x) = (1 + x)^n C(0, 0) \\ &= (1 + x)^n \end{aligned}$$

Recurrence with Two Indices

Ex: Let $f(n, r)$ be the # of r-combinations of n distinct objects, with unlimited supply.

Find $f(n, r)$

A: By considering whether we select the first object or not, we get the recurrence below :

$$f(n, r) = f(n, r-1) + f(n-1, r)$$

where $f(n, 0) = 1$ for $n \geq 0$, $f(0, r) = 0$ for $r > 0$

Recurrence with Two Indices

- Next, we define a GF $F_n(x)$:

$$F_n(x) = f(n,0) + f(n,1)\,x + f(n,2)\,x^2 + \ldots$$

- From the previous recurrence, we get:

$$\sum_{r=1}^{\infty} f(n, r)\, x^r = \sum_{r=1}^{\infty} \big(f(n, r-1) + f(n-1, r)\big)\, x^r$$

Recurrence with Two Indices

- Thus we have

$$F_n(x) - f(n, 0) = x\, F_n(x) + F_{n-1}(x) - f(n-1, 0)$$

- By simplifying terms, we get :

$$\begin{aligned} F_n(x) &= (1-x)^{-1} F_{n-1}(x) = \ldots \\ &= (1-x)^{-n} F_0(x) = (1-x)^{-n} f(0, 0) \\ &= (1-x)^{-n} \end{aligned}$$

Recurrence with Two Indices

Ex: Find the # of n-bit binary strings with exactly r pairs of adjacent 1's and no adjacent 0's.

E.g., two pairs of adjacent 1's in 111

A: Let $a_{n,r}$ = # of such strings

Let $b_{n,r}$ = # of such strings that end with 1

Let $c_{n,r}$ = # of such strings that end with 0

Clearly,

$$a_{n,r} = b_{n,r} + c_{n,r}$$

Recurrence with Two Indices

- Since an n-bit string that has r pairs of 1's, no adjacent 0's, and a 1 as the last digit can be formed by appending a 1 either to
  1. An $(n-1)$-bit string that has $r-1$ pairs of 1's, no adjacent 0's, and a 1 as the last digit ; or
  2. An $(n-1)$-bit string that has r pairs of 1's, no adjacent 0's, and a 0 as the last digit.

We get :

$$b_{n,r} = b_{n-1,r-1} + c_{n-1,r}$$

Recurrence with Two Indices

- Similarly, we get :

$$c_{n,r} = b_{n-1,r}$$

- Combining the two results, we get :

$$b_{n,r} = b_{n-1,r-1} + b_{n-2,r}$$

- What are the boundary conditions ??

Recurrence with Two Indices

- It is easy to check that

$$b_{n,0} = 1 \qquad \text{for } n \geq 1$$
$$b_{i,j} = 0 \qquad \text{for } i \leq j$$

and we set $b_{0,0} = 1$ for convenience

- Let

$$B_n(x) = b_{n,0} + b_{n,1}\, x + b_{n,2}\, x^2 + \ldots$$

Recurrence with Two Indices

- From the previous recurrence, we get:

$$\sum_{r=1}^{\infty} b_{n,r}\, x^r = \sum_{r=1}^{\infty} \left(b_{n-1,r-1} + b_{n-2,r}\right) x^r$$

- Thus for $n \geq 3$, we have :

$$B_n(x) \;=\; x\, B_{n-1}(x) + B_{n-2}(x)$$

Recurrence with Two Indices

- As for $B_0(x), B_1(x), B_2(x)$, we have :

$$B_0(x) = b_{0,0} = 1$$
$$B_1(x) = b_{1,0} + b_{1,1}x = 1$$
$$B_2(x) = b_{2,0} + b_{2,1}x + b_{2,2}x^2 = 1 + x$$

- Next, we let

$$\mathcal{B}(y, x) = B_0(x) + B_1(x)\,y + B_2(x)\,y^2 + \ldots$$

Recurrence with Two Indices

- Then from the recurrence of $B_n(x)$, we get:

$$\sum_{n=3}^{\infty} B_n(x)\, y^n = \sum_{n=3}^{\infty} \left(x\, B_{n-1}(x) + B_{n-2}(x) \right) y^n$$

- Consequently, we have :

$$\text{LHS} = \mathcal{B}(y, x) - (1 + x)\, y^2 - y - 1$$

$$\text{RHS} = x\, y \left(\mathcal{B}(y, x) - y - 1 \right) + y^2 \left(\mathcal{B}(y, x) - 1 \right)$$

Recurrence with Two Indices

- By re-arranging terms, we obtain :

$$\mathcal{B}(y, x) = \left(1 + (1 - x)y \right) \left(1 - x\,y - y^2\right)^{-1}$$

$$= 1 + y + (1 + x)\,y^2 + (1 + x + x^2)\,y^3$$

$$+ (1 + 2x + x^2 + x^3)\,y^4$$

$$+ (1 + 2x + 3x^2 + x^3 + x^4)\,y^5 + \ldots$$

Recurrence with Two Indices

- Next, from the recurrence of $c_{n,r}$, we get :

$$\mathcal{C}(y, x) = y\ \mathcal{B}(y, x)$$

- Consequently :

$$\begin{aligned}\mathcal{A}(y, x) &= \mathcal{B}(y, x) + \mathcal{C}(y, x) = (1 + y)\ \mathcal{B}(y, x) \\ &= 1 + 2y + (2 + x)\ y^2 + (2 + 2x + x^2)\ y^3 \\ &\quad + (2 + 3x + 2x^2 + x^3)\ y^4 \\ &\quad + (2 + 4x + 4x^2 + 2x^3 + x^4)\ y^5 + \ldots\end{aligned}$$