Solution of Assignment 6

Wisely

Question 1

Suppose $(G, \bigstar)$ is a group

Also $(H, \bigstar)$ and $(K, \bigstar)$ are subgroups of $(G, \bigstar)$

Show that $(H \cap K, \bigstar)$ is a subgroup of $(G, \bigstar)$

Solution of Question 1

Our target is to show :

(1) ★ is a closed operation in $H \cap K$

(2) There exists an identity element *e*' in $H \cap K$

(3) For each element *a*, the inverse of *a* is in $H \cap K$

Closed:

Let *a*, *b* be any two elements in $H \cap K$

Then *a* ★ *b* is in *H* (why?)

Also, *a* ★ *b* is in *K* (why?)

➔ *a* ★ *b* is in $H \cap K$

Solution of Question 1

Identity :

Let e = identity of (G, ★)

Since (H, ★) is subgroup of (G, ★), e must be in H

Also (K, ★) is subgroup of (G, ★), e must be in K

➔ e is in $H \cap K$, and it is easy to check that for any element a in $H \cap K$, e ★ $a = a$ ★ $e = a$

Inverse:

For any element a in $H \cap K$,

inverse of a is in H and inverse of a is in K (why?)

➔ Such inverse a^{-1} must be in $H \cap K$ and it is easy to check that a^{-1} ★ $a = a$ ★ $a^{-1} = e$

Question 2

- Suppose $(G, \bigstar)$ is a group
  and e be its identity element

- For each element g in G, we define
  $ord(g)$ = min positive k such that $g^k = e$

- Show that $ord(g)$ must divide $|\ G\ |$

Solution of Question 2

- Our target is to construct the set

  $$L = \{ g^1, g^2, \ldots, g^k \} \text{ , where } k = ord(g)$$

  and show that $(L, \bigstar)$ is a subgroup of $(G, \bigstar)$

  Then by Lagrange Theorem, k must divide $|G|$

- To show that $(L, \bigstar)$ is a subgroup, it is easy to see that

  (1) Closed : $g^i \bigstar g^j = g^{i+j \text{ (rem } k)}$ which is in L

  (2) Identity exists (by definition $g^k = e$)

  (3) For each g^j, g^{k-j} is in L such that $g^j \bigstar g^{k-j} = e$

  ➔ inverse exists for each element

Question 3

- A rod is divided into six segments, and each segment will be colored by one of the *n* colors
- Two colorings are the same if one can be transformed to the other by 180° rotation

- How many distinct colorings ?

Solution of Question 3

Let S be the set of all n^6 colorings.

Let $(G, \circ)$ be permutation group such that each permutation in G correspond to a possible mapping of a coloring to another due to a series of rotations.

➔ G has two elements:

Identity, rotation 180°

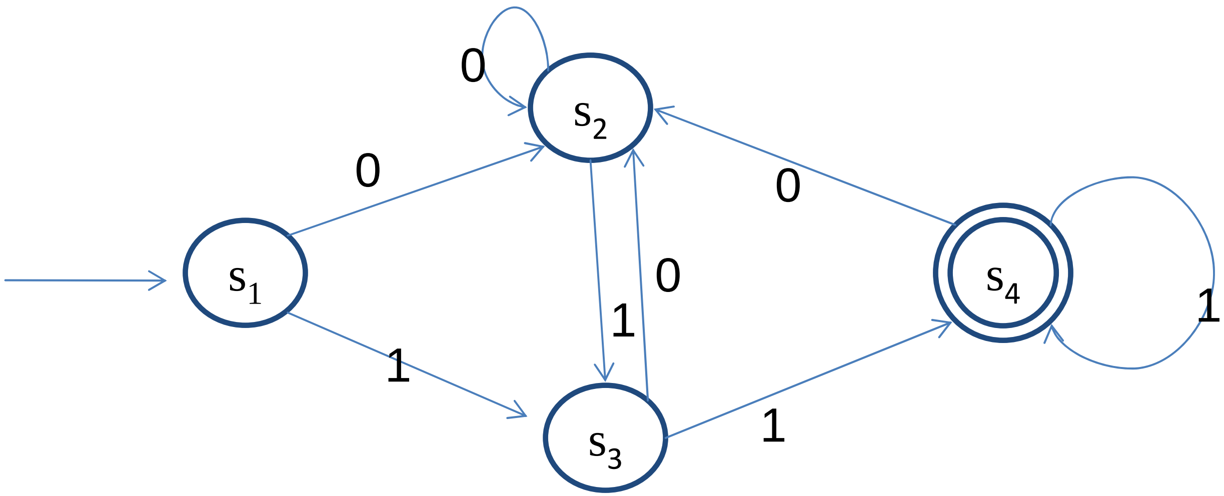
Solution of Question 3

- To find the number of distinct colorings, it is the same to as to find out how many equivalence classes obtained by the relation induced by $(G, \circ)$
- By Burnside's Theorem, the number of classes would be:

$$(n^6 + n^3) / 2$$

Question 4

- Design a finite automaton that accepts exactly all binary strings each of which ends with 11



Question 5

- A *palindrome* is a string that reads the same forward and backward

- Show that the language

$$L = \{ w \mid w \text{ is a binary palindrome} \}$$

is non-regular.

Solution of Question 5

Proof (By pumping lemma) :

Assume that L is regular.

➔ Let p be the pumping length

➔ First, we see that the string $s = 0^p10^p$ is in L

➔ Thus by pumping lemma, there is a way to divide s into $s = xyz$ satisfying

(1) for any $i \geqq 0$, the string xy^iz is in L

(2) $|y|>0$ and (3) $|xy| \leqq p$

Solution of Question 5

Proof (cont) :

- ➔ If the above is true, then condition 3 implies that *y* would contain only 0s
- ➔ Then the string *xyyz* must not be a palindrome

  (since more 0s before 1 than after 1)

  Thus, contradiction.
- ➔ Therefore *L* is not regular

Question 6

- Suppose (G, ★) is a group, and both (H, ★) and (K, ★) are subgroup of (G, ★)
- Define

$$HK = \{\ hk \mid h \text{ in } H \text{ and } k \text{ in } K\ \}$$

and define KH similarly.

- Show that (HK, ★) is a subgroup of (G, ★) if and only if HK=KH.

Solution of Question 6

Proof (if case):

Our target is to show that $HK \subseteq KH$ and $KH \subseteq HK$

This will then imply $KH = HK$

- First, suppose x is in HK

  Then the inverse of x must be in HK

  (since HK is a subgroup)

  ➔ Say $x^{-1} = hk$ for some h in H and k in K

  ➔ Then, $x = k^{-1}h^{-1}$ so that x is in KH

Solution of Question 6

Proof (if case):

- Next, suppose x is in KH
  - ➔ Say $x^{-1} = kh$ for some h in H and k in K
  - ➔ Then the inverse of x is $h^{-1}k^{-1}$
    which is in HK (since h^{-1} in H and k^{-1} in K)
  - ➔ Then, x is in HK (since HK is a group)

Solution of Question 6

Proof (only if case):

We want to show that

if $HK = KH$, then $(HK, \bigstar)$ is a group

Our target is to show :

(1) $\bigstar$ is closed

Let $x = h_0k_0 = k_1h_1$ and $y = h_2k_2 = k_3h_3$

➔ $x \bigstar y = h_0k_0k_3h_3 = h_0k_4h_3 = h_0h_5k_5 = h_6k_5$

➔ $x \bigstar y$ is in HK

(2) Identity exists (easy)

(3) Inverse exists (easy)

Question 7

- Let D be the set of binary strings where each string has equal # of substrings 01 and 10

- Show that D is a regular language