Solution of Assignment 6

Wisely
Question 1

Suppose \((G, \star)\) is a group
Also \((H, \star)\) and \((K, \star)\) are subgroups of \((G, \star)\)

Show that \((H \cap K, \star)\) is a subgroup of \((G, \star)\)
Solution of Question 1

Our target is to show:

(1) ★ is a closed operation in \( H \cap K \)
(2) There exists an identity element \( e' \) in \( H \cap K \)
(3) For each element \( a \), the inverse of \( a \) is in \( H \cap K \)

Closed:

Let \( a, b \) be any two elements in \( H \cap K \)
Then \( a \star b \) is in \( H \) (why?)
Also, \( a \star b \) is in \( K \) (why?)
\( \Rightarrow \) \( a \star b \) is in \( H \cap K \)
Solution of Question 1

Identity:

Let \( e = \text{identity of } (G, \star) \)

Since \( (H, \star) \) is subgroup of \( (G, \star) \), \( e \) must be in \( H \)

Also \( (K, \star) \) is subgroup of \( (G, \star) \), \( e \) must be in \( K \)

\( \Rightarrow \) \( e \) is in \( H \cap K \), and it is easy to check that for any element \( a \) in \( H \cap K \), \( e \star a = a \star e = a \)

Inverse:

For any element \( a \) in \( H \cap K \),

inverse of \( a \) is in \( H \) and inverse of \( a \) is in \( K \) (why?)

\( \Rightarrow \) Such inverse \( a^{-1} \) must be in \( H \cap K \) and it is easy to check that \( a^{-1} \star a = a \star a^{-1} = e \)
Question 2

• Suppose \((G, \star)\) is a group and \(e\) be its identity element

• For each element \(g\) in \(G\), we define
  \[
  \text{ord}(g) = \min \text{ positive } k \text{ such that } g^k = e
  \]

• Show that \(\text{ord}(g)\) must divide \(|G|\)
Solution of Question 2

• Our target is to construct the set

$$L=\{g^1, g^2, ..., g^k\}, \text{ where } k = \text{ord}(g)$$

and show that $$(L, \star)$$ is a subgroup of $$(G, \star)$$

Then by Lagrange Theorem, $k$ must divide $|G|$

• To show that $$(L, \star)$$ is a subgroup, it is easy to see that

1. Closed: $$g^i \star g^j = g^{i+j \text{(rem } k)}$$ which is in $L$
2. Identity exists (by definition $$g^k = e$$)
3. For each $$g^j$$, $$g^{k-j}$$ is in $L$ such that $$g^j \star g^{k-j} = e$$
   $$\Rightarrow$$ inverse exists for each element
Question 3

• A rod is divided into six segments, and each segment will be colored by one of the $n$ colors
• Two colorings are the same if one can be transformed to the other by $180^\circ$ rotation

• How many distinct colorings?
Solution of Question 3

Let $S$ be the set of all $n^6$ colorings.

Let $(G, \circ)$ be permutation group such that each permutation in $G$ correspond to a possible mapping of a coloring to another due to a series of rotations.

$\Rightarrow G$ has two elements:

- Identity, rotation $180^\circ$
Solution of Question 3

• To find the number of distinct colorings, it is the same to as to find out how many equivalence classes obtained by the relation induced by \((G, \circ)\).

• By Burnside’s Theorem, the number of classes would be:

\[
\frac{n^6 + n^3}{2}
\]
Question 4

- Design a finite automaton that accepts exactly all binary strings each of which ends with 11
Question 5

• A palindrome is a string that reads the same forward and backward

• Show that the language

\[ L = \{ w \mid w \text{ is a binary palindrome} \} \]

is non-regular.
Solution of Question 5

Proof (By pumping lemma) :

Assume that $L$ is regular.

- Let $p$ be the pumping length
- First, we see that the string $s = 0^p10^p$ is in $L$
- Thus by pumping lemma, there is a way to divide $s$ into $s = xyz$ satisfying
  (1) for any $i \geq 0$, the string $xy^iz$ is in $L$
  (2) $|y| > 0$ and (3) $|xy| \leq p$
Solution of Question 5

Proof (cont) :

⇒ If the above is true, then condition 3 implies that $y$ would contain only 0s
⇒ Then the string $xyyz$ must not be a palindrome (since more 0s before 1 than after 1)
   Thus, contradiction.
⇒ Therefore $L$ is not regular
Question 6

• Suppose \((G, \star)\) is a group, and both \((H, \star)\) and \((K, \star)\) are subgroup of \((G, \star)\)

• Define

\[
HK = \{ \, hk \mid h \text{ in } H \text{ and } k \text{ in } K \, \}
\]

and define \(KH\) similarly.

• Show that \((HK, \star)\) is a subgroup of \((G, \star)\) if and only if \(HK=KH\).
Solution of Question 6

Proof (if case):

Our target is to show that $HK \subseteq KH$ and $KH \subseteq HK$
This will then imply $KH = HK$

• First, suppose $x$ is in $HK$
Then the inverse of $x$ must be in $HK$
   (since $HK$ is a subgroup)
   ➔ Say $x^{-1} = hk$ for some $h$ in $H$ and $k$ in $K$
   ➔ Then, $x = k^{-1}h^{-1}$ so that $x$ is in $KH$
Solution of Question 6

Proof (if case):

• Next, suppose $x$ is in $KH$
  ➞ Say $x^{-1} = kh$ for some $h$ in $H$ and $k$ in $K$
  ➞ Then the inverse of $x$ is $h^{-1}k^{-1}$
    which is in $HK$ (since $h^{-1}$ in $H$ and $k^{-1}$ in $K$)
  ➞ Then, $x$ is in $HK$ (since $HK$ is a group)
Solution of Question 6

Proof (only if case):

We want to show that

if \( HK = KH \), then \((HK, \star)\) is a group

Our target is to show:

1. \( \star \) is closed

   Let \( x = h_0k_0 = k_1h_1 \) and \( y = h_2k_2 = k_3h_3 \)

   \[ x \star y = h_0k_0k_3h_3 = h_0k_4h_3 = h_0h_5k_5 = h_6k_5 \]

   \( x \star y \) is in \( HK \)

2. Identity exists (easy)

3. Inverse exists (easy)
Question 7

- Let $D$ be the set of binary strings where each string has equal # of substrings 01 and 10

- Show that $D$ is a regular language