Assignment 5

Speaker: Wisely

Question 1

- Suppose $F(x)$ is a polynomial such that all the coefficients are integers

  Also $F(0) = F(1) = 1$

- Show that F does not have any integral root.

  That is, no integer z such that $F(z) = 0$

Hint of Question 1

- Solve it directly.

Question 2

- Show that if n is an odd number, then

$$
\begin{aligned}
& 1 \times 3 \times 5 \times \ldots \times (2n - 1) \\
+ \quad & 2 \times 4 \times 6 \times \ldots \times (2n)
\end{aligned}
$$

is a multiple of $2n + 1$

Hint of Question 2

- Solve it directly.

Question 3

- Let p be a prime

- Show that if there exists n such that

$$n^2 \equiv -1 \pmod{p} ,$$

then $p \not\equiv 3 \pmod{4}$

Hint of Question 3

- Fermat's Little Theorem

Question 4

- Let p be a prime
- Let a and b be two integers coprime to p

- Show that

$$ax \equiv b \qquad (\mathrm{mod}\ p)$$

if and only if

$$x \equiv a^{p-2}b \qquad (\mathrm{mod}\ p)$$

Hint of Question 4

- Fermat’s Little Theorem
- Existence and Uniqueness

Question 5

- Prove that if

$$n^j \equiv 1 \pmod{m} \quad \text{and} \quad n^k \equiv 1 \pmod{m},$$

then $\quad n^{\gcd(j,k)} \equiv 1 \pmod{m}$

Hint of Question 5

- Property of GCD.

Question 6

- Prove that $\varphi(n^m) = n^{m-1}\varphi(m)$.

Hint of Question 6

- Solve it directly.

Question 7

- Compute $\varphi(999)$.

Hint of Question 7

- Solve it directly.

Question 8

- n is a perfect number if the sum of all the proper divisors of n is exactly n

- Example:

$$6 = 1 + 2 + 3 = 6$$

$$28 = 1 + 2 + 4 + 7 + 14 = 28$$

Question 8

Theorem 1 (By Euler).

An even number n is a perfect number if and only if

$$n = 2^m(2^{m+1} - 1) \text{ and } 2^{m+1} - 1 \text{ is prime}$$

- Show that Theorem 1 is correct

Hint of Question 8

- Solve it step by step.

Question 9

- Show that for all $n > 1$,

$$2^n \not\equiv 1 \pmod{n}$$

Hint of Question 9

- Challenging (No hints)