Assignment 5

Speaker: Wisely
Question 1

• Suppose $F(x)$ is a polynomial such that all the coefficients are integers

Also $F(0) = F(1) = 1$

• Show that $F$ does not have any integral root. That is, no integer $z$ such that $F(z) = 0$
Hint of Question 1

• Solve it directly.
Question 2

• Show that if $n$ is an odd number, then

\[ 1 \times 3 \times 5 \times \ldots \times (2n - 1) + 2 \times 4 \times 6 \times \ldots \times (2n) \]

is a multiple of $2n + 1$
Hint of Question 2

• Solve it directly.
Question 3

• Let $p$ be a prime

• Show that if there exists $n$ such that

$$n^2 \equiv -1 \pmod{p},$$

then $p \not\equiv 3 \pmod{4}$
Hint of Question 3

• Fermat's Little Theorem
Question 4

• Let $p$ be a prime
• Let $a$ and $b$ be two integers coprime to $p$
• Show that
  \[ ax \equiv b \pmod{p} \]
  if and only if
  \[ x \equiv a^{p-2}b \pmod{p} \]
Hint of Question 4

• Fermat’s Little Theorem
• Existence and Uniqueness
Question 5

• Prove that if

\[ n^j \equiv 1 \pmod{m} \quad \text{and} \quad n^k \equiv 1 \pmod{m}, \]

then \[ n^{\gcd(j, k)} \equiv 1 \pmod{m} \]
Hint of Question 5

• Property of GCD.
Question 6

• Prove that $\varphi(n^m) = n^{m-1}\varphi(m)$. 
Hint of Question 6

• Solve it directly.
Question 7

• Compute $\varphi(999)$.
Hint of Question 7

• Solve it directly.
Question 8

• *n* is a **perfect number** if the sum of all the proper divisors of *n* is exactly *n*

• Example:

\[ 6 = 1 + 2 + 3 = 6 \]
\[ 28 = 1 + 2 + 4 + 7 + 14 = 28 \]
Question 8

Theorem 1 (By Euler).

An even number $n$ is a perfect number if and only if

$$n = 2^m(2^{m+1} - 1) \text{ and } 2^{m+1} - 1 \text{ is prime}$$

• Show that Theorem 1 is correct
Hint of Question 8

• Solve it step by step.
Question 9

• Show that for all $n > 1$,

$$2^n \equiv 1 \pmod{n}$$
Hint of Question 9

• Challenging (No hints)