Assignment 4

Speaker: Wisely

Question 1

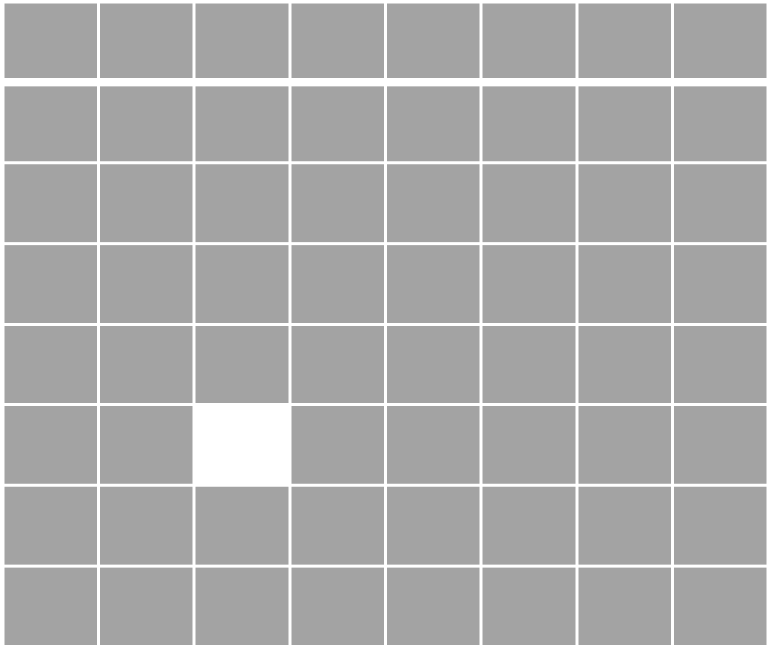
- Let S be a 2×2 square
- If we remove a 1×1 square from S, we call the resulting figure an L-shaped tile :

Question 1

- Consider a figure formed by removing any 1×1 square inside a $2^n \times 2^n$ square.
- Show that the figure can be covered by non-overlapping L-shaped tiles

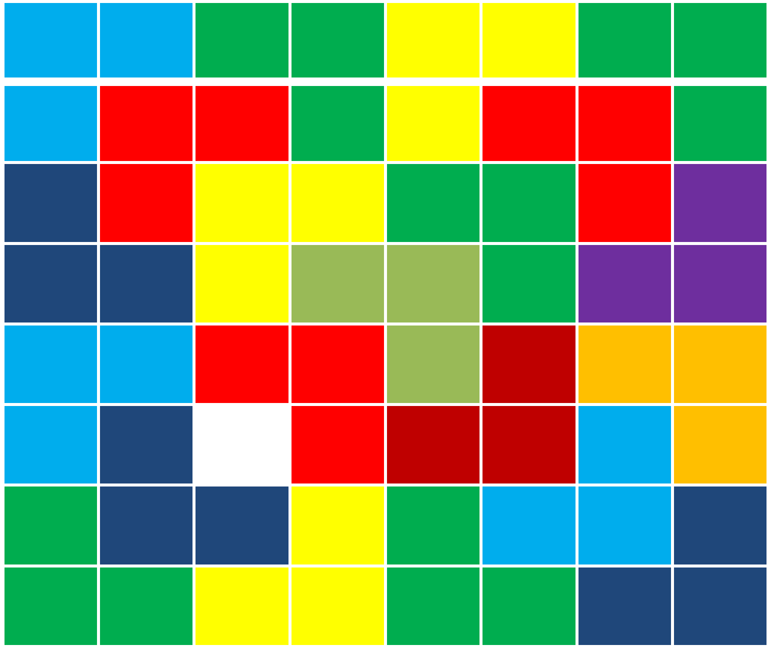
Example for Question 1

8×8 square with a 1×1 square removed



Example for Question 1

Covering with L-shaped tiles



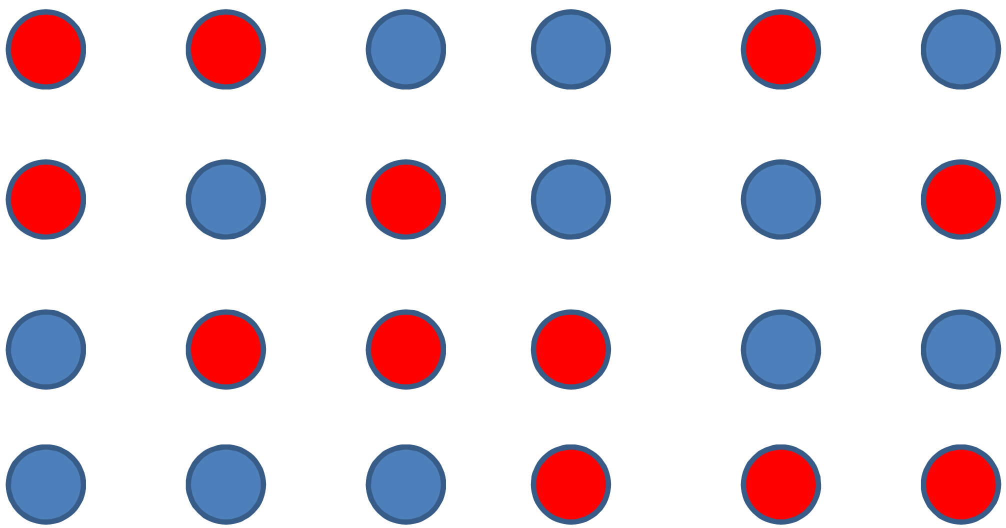
Hint of Question 1

- By induction

Question 2

- Consider a $2m \times 2n$ grids, with
  - Each grid point is either colored red or blue
  - For any row or column,

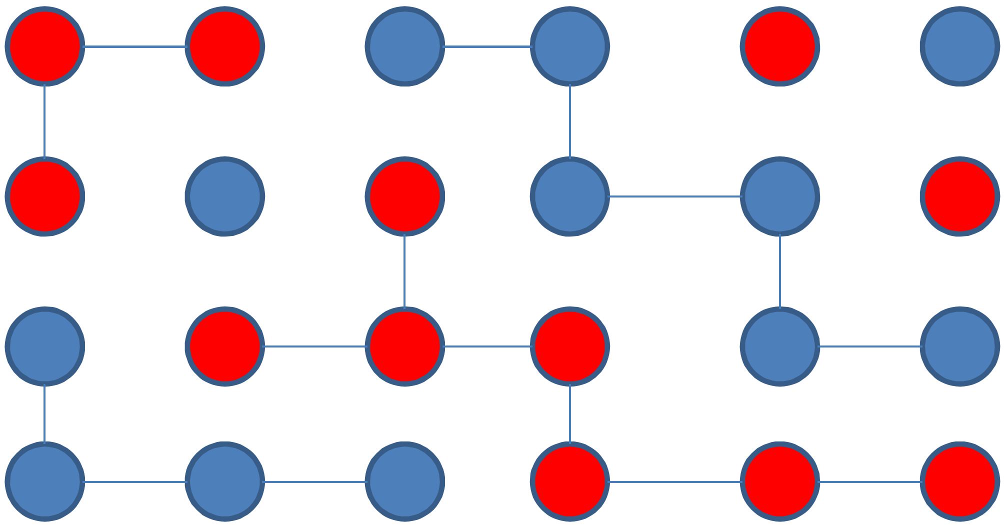
$$\text{\# red points} = \text{\# blue points}$$



Question 2

- When two adjacent points have the same color, we join them by a line of that color
- Show that no matter how the points are colored,

red lines = # blue lines



Hint of Question 2

- By induction

Question 3

- Give a sequence of n integers $a_1, a_2, a_3, \ldots, a_n$
- Show that there exists a contiguous subsequence whose sum is divisible by n.

Hint of Question 3

- Pigeonhole principle

Question 4

- Suppose $n + 1$ integers are chosen from 1 to $2n$
- Show that there exist two of the chosen numbers which are relatively prime.

Hint of Question 4

- Pigeonhole principle

Question 5

- There are 100 people at a party.
- Each has even number (possible 0) of friends
- Prove that we can always find three people with the same number of friends

Hint of Question 5

- Before solving Question 5, let us solve the following simpler question first :
  - Given 100 people in a party, each person may have any number of friends
  - Show that at least 2 people will have the same number of friends

Proof :

There are two cases.

Case 1: Nobody has 0 friends

➔ # friends for each person is from 1 to 99

➔ By pigeonhole principle, at least two persons have same # friends

Case 2: Someone has 0 friends

➔ # friends for each person is from 0 to 98 (why?)

Hint of Question 5

- Pigeonhole principle

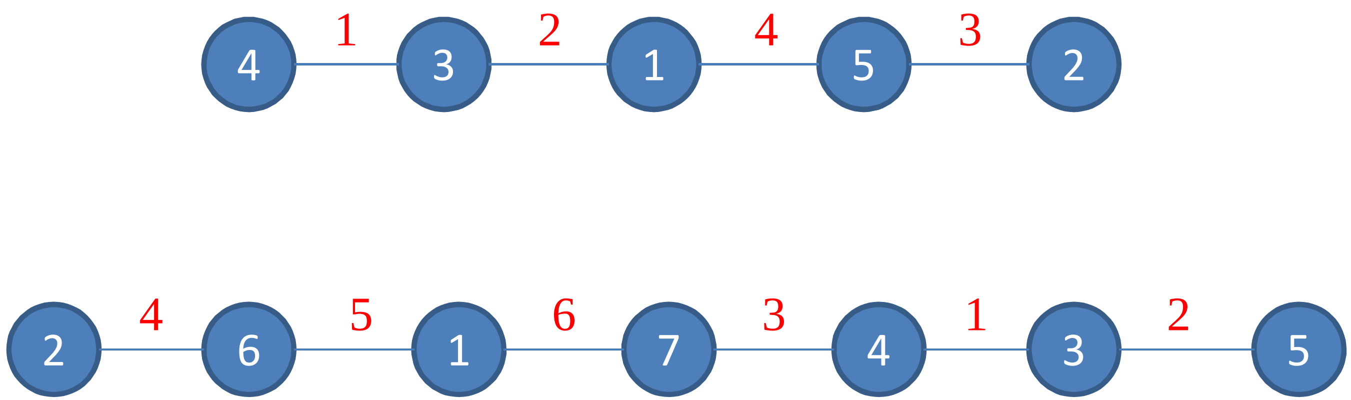
Question 6

- Let $P = (v_1, v_2, \ldots, v_n)$ be a path with n vertices
- Show that we can assign each vertex v_i a distinct integer $f(i)$ chosen from 1 to n, such that

$$\left| f(i) - f(i+1) \right|$$

for $i = 1, 2, \ldots, n-1$ are all distinct

Example of Question 6



Hint of Question 6

- Prove it by construction
  - Find a simple strategy to assign the values to the vertices that works for each *n*

- Prove by induction (alternative)