Solutions of Assignment 4

Speaker: Wisely

Question 1

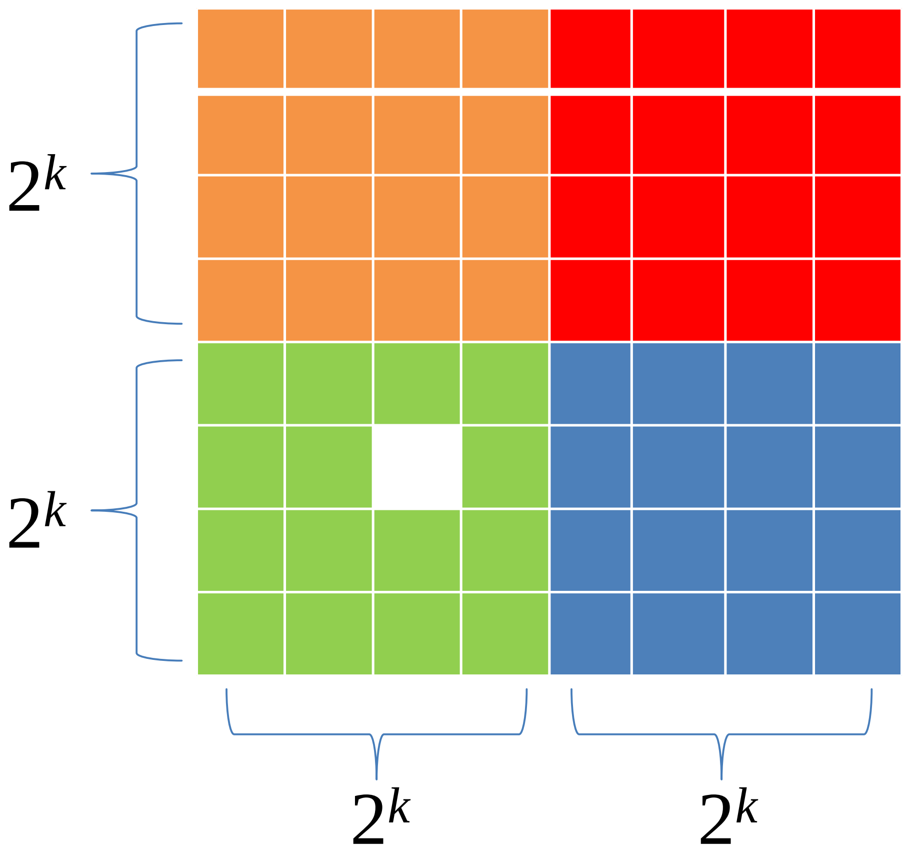
- Consider a figure formed by removing any 1×1 square inside a $2^n \times 2^n$ square.
- Show that the figure can be covered by non-overlapping L-shaped tiles

Proof (By Induction):

Base case (n=1): Trivial.

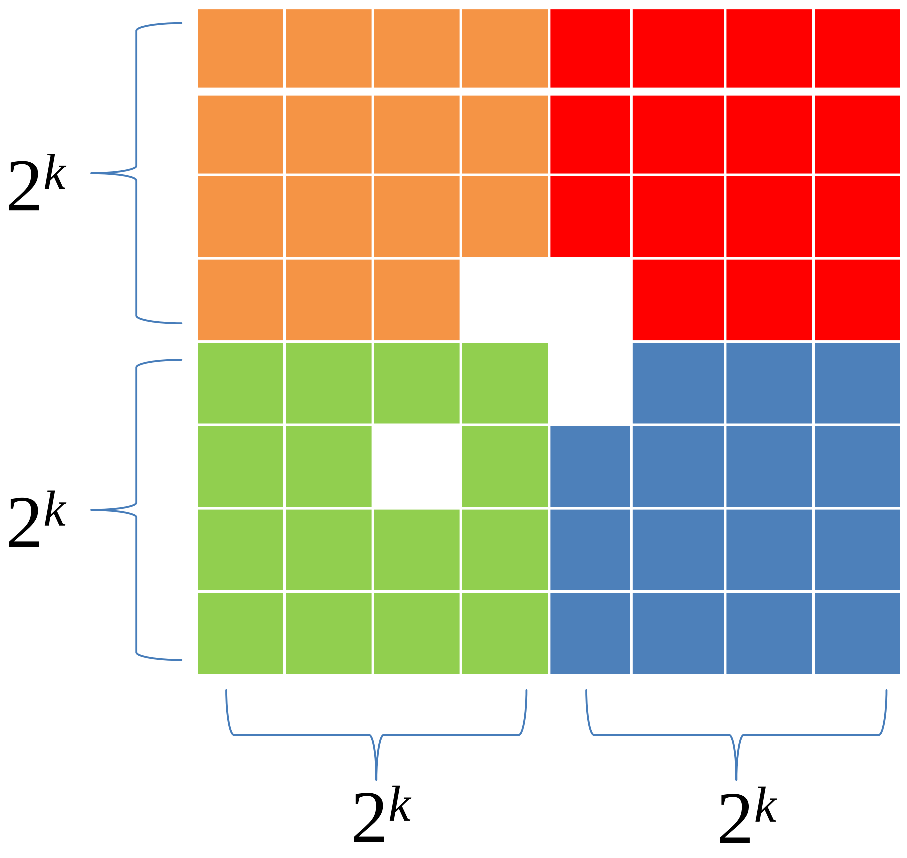
Inductive case (assume $n=k$ is true): Consider $n=k+1$.

We separate $2^{k+1} \times 2^{k+1}$ square into four squares as below



Proof (cont):

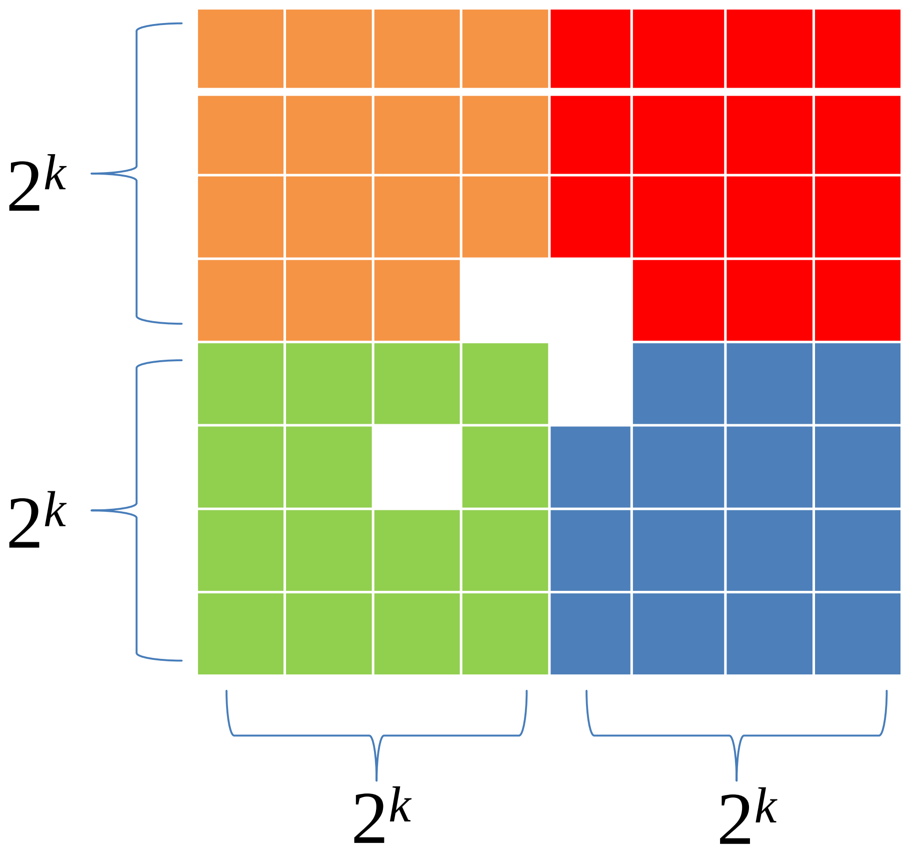
For those three $2^k \times 2^k$ squares that do not contain the removed square, we select a corner from each of them, and cover them by an L-shaped tile as shown below



Proof (cont):

➔ By the inductive hypothesis,
each of them can now be covered by L-shaped tiles

➔ This completes the proof



Question 2

- Consider a $2m \times 2n$ grid, with
  - Each grid point is either colored red or blue
  - For any row or column,

    # red points = # blue points
- When two adjacent points have the same color, we join them by a line of that color
- Show that no matter how the points are colored,

  # red lines = # blue lines

Proof (By Induction):

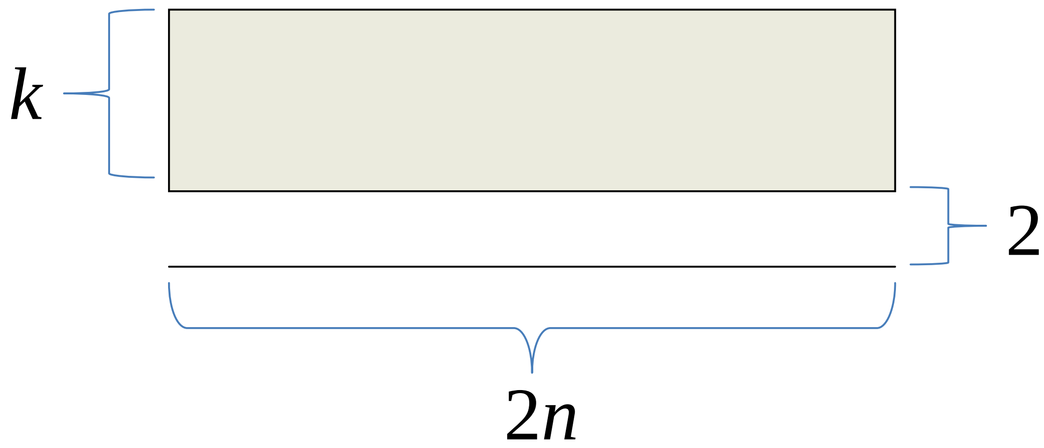
We prove that for any m,

vertical red lines = # vertical blue lines

in any $m \times 2n$ grid

Base case (m=1): Trivial, since there are no lines

Inductive Case (assume that m=k is true) :



Proof (cont):

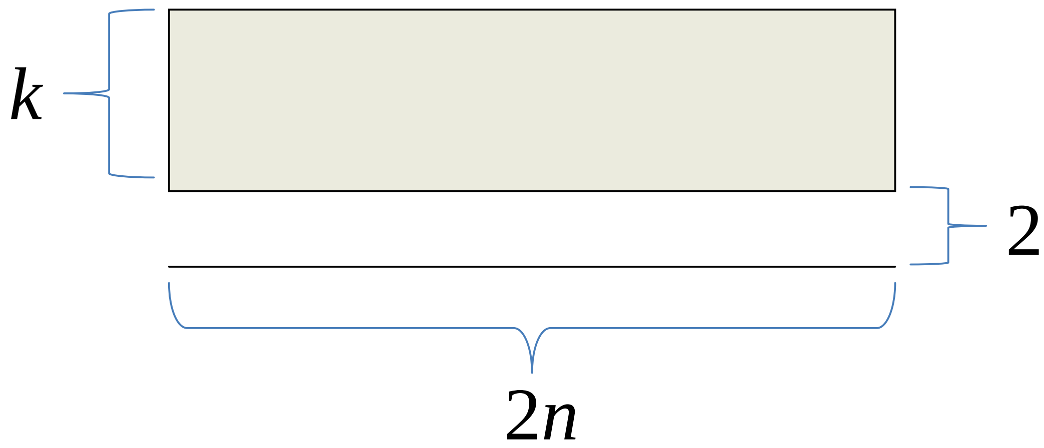
Each of the last two rows has n red and n blue points

Let x = # of vertical pairs with different colors

➔ Thus, there are $(2n - x)/2$ vertical red pairs (lines)

and there are $(2n - x)/2$ vertical blue pairs (lines)

➔ Inductive case is true



Proof (cont):

Similarly, we can show that for any n,

horizontal red lines = # horizontal blue lines

in any $2m \times n$ grid

In conclusion, in any $2m \times 2n$,

red lines = # blue lines

Question 3

- Give a sequence of n integers $a_1, a_2, a_3, \ldots, a_n$
- Show that there exists a contiguous subsequence whose sum is divisible by n.

Proof:

Let $S_i = a_1+a_2+...+a_i$, where $1 \leqq i \leqq n$

Let $r_i = S_i$ rem n

There are two cases.

Case 1: Some $r_i = 0$

➔ $a_1, a_2, \ldots, a_i$ is the desired subsequence

Case 2: All r_i are between 1 and $n - 1$

➔ By Pigeonhole Principle,

there must exist $r_i = r_j$, for some $j > i$.

➔ $a_i, a_{i+1}, \ldots, a_j$ is the desired subsequence

Question 4

- Suppose $n + 1$ integers are chosen from 1 to $2n$
- Show that there exist two of the chosen numbers which are relatively prime

Proof:

We partition the $2n$ integers into n boxes as follows :

(1,2) (3,4) … … $(2n\text{-}3,2n\text{-}2)$ $(2n\text{-}1,2n)$

By Pigeonhole Principle,
when we choose $n + 1$ integers,
two integers must be chosen from the same box

➔ These two chosen integers are relatively prime

Question 5

- There are 100 people at a party.
- Each has even number (possible 0) of friends
- Prove that we can always find three people with the same number of friends

Proof: There are three cases.

Case 1: At most 1 person has 0 friends

➔ # friends for each remaining person is from 2 to 98

➔ By Generalized Pigeonhole Principle,
at least three persons have same # friends

Case 2: Exactly two persons has 0 friends

➔ # friends for each remaining person is from 2 to 96

Case 3: At least three persons has 0 friends

➔ Trivial

Question 6

- Let $P = (v_1, v_2, \ldots, v_n)$ be a path with n vertices
- Show that we can assign each vertex v_i a distinct integer $f(i)$ chosen from 1 to n, such that

$$| \ f(i) - f(i+1) \ |$$

for $i = 1, 2, \ldots, n - 1$ are all distinct

Proof :

We assign each vertex a value in the following way :