CS2335 Special Topics in Discrete Structure

Homework 4

Due: 1:10 pm, December 3, 2009 (before class)

1. Let $\square$ denote a 1×1 square. Let S be a 2×2 square formed by 2×2 pieces of $\square$s. If we remove any $\square$ from S, we call the resulting figure an L-shaped tile (See Figure 1(a)).

   Next, consider a figure formed by removing any one of the $\square$ inside a $2^n \times 2^n$ square (See Figure 1(b)). Show that the figure can be covered by non-overlapping L-shaped tiles.

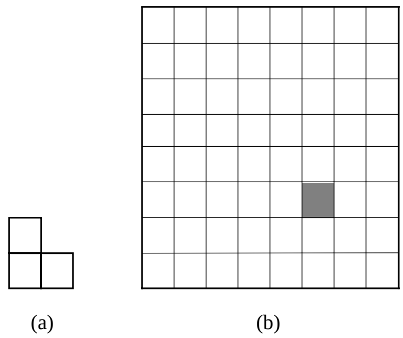

Figure 1: (a) An L-shaped tile. (b) A 8×8 square with one piece missing.

2. Consider a $2m \times 2n$ grids, so that each grid point is either colored red or blue, with number of red points is equal to number of blue points, for any row and any column. When two adjacent points (up, down, left, right) have the same color c, we join them by a line of color c (See Figure 2 for an example). Show that no matter how the original points are colored, the number of red lines must be equal to the number of blue lines.

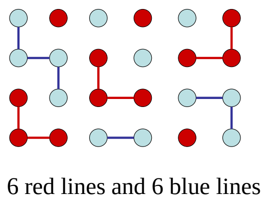

Figure 2: Number of red lines and blue lines are the same.

3. Give a sequence of n integers $a_1, a_2, a_3, ..., a_n$, show that there exists a contiguous subsequence whose sum is divisible by n.

4. Suppose $n + 1$ integers are chosen from 1 to $2n$. Show that there exist two of the chosen numbers which are relatively prime.

5. There are 100 people at a party. Each person has an even number (possible zero) of friends. Prove that we can always find three people with the same number of friends.

6. Suppose we have a path P with n vertices, $P = (v_1, v_2, \ldots, v_n)$. Show that we can assign each vertex v_i a distinct integer $f(i)$ chosen from 1 to n, such that $|f(i) - f(i + 1)|$ for $i = 1, 2, \ldots n - 1$ are all distinct.

   For instance, when $n = 5$, the following is a possible assignment satisfying the above requirement: $f(1) = 2$, $f(2) = 5$, $f(3) = 1$, $f(4) = 3$, $f(5) = 4$.