Assignment 3

Speaker: Wisely

Question 1

- Solve the following recurrence relations.

(a) $a_n = 5a_{n-1} + 6a_{n-2}, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 3$

(b) $2a_{n+2} - 11a_{n+1} + 5a_n = 0, \quad n \geq 0, \quad a_0 = 2, \quad a_1 = -8$

(c) $a_n - 6a_{n-1} + 9a_{n-2} = 0, \quad n \geq 2, \quad a_0 = 5, \quad a_1 = 12$

Solution :

(a) $a_n = (3/7)(-1)^n + (4/7)6^n, \quad n \geqq 0$

(b) $a_n = 4(1/2)^n - 2(5)^n, \quad n \geqq 0$

(c) $a_n = (5 - n)3^n, \quad n \geqq 0$

Question 2

Solve the following recurrence relations.

(a) $a_{n+1} - a_n = 2n + 3, \quad n \geq 0, \quad a_0 = 1$

(b) $a_{n+1} - a_n = 2n^2 - n, \quad n \geq 0, \quad a_0 = 3$

(c) $a_{n+1} - 2a_n = 2^n, \quad n \geq 0, \quad a_0 = 1$

Solution :

(a) $a^{(p)} = n^2+2n+1$, $a^{(h)} = 1^n$

$a_n = (n+1)^2, \qquad n \geqq 0$

(b) $a^{(p)} = (2/3)n^3 - (3/2)n^2 + (5/6)n + 3, \quad a^{(h)} = 1^n$

$a_n = (2/3)n^3 - (3/2)n^2 + (5/6)n + 3, \qquad n \geqq 0$

(c) $a^{(p)} = n2^{(n-1)}, \qquad a^{(h)} = 2^n$

$a_n = 2^n + n2^{(n-1)}, \quad n \geqq 0$

Question 3

- Solve the following recurrence relations by the method of generating functions

(a) $a_{n+2} - 2a_{n+1} + a_n = 2^n,\ n \ge 0, \quad a_0 = 1,\ a_1 = 2$

(b) $a_{n+1} = 2a_n - b_n + 2, \qquad n \ge 0,$

$b_{n+1} = -a_n + 2b_n - 1, \qquad n \ge 0,$

$a_0 = 0,\ \ b_0 = 1$

Solution :

(a) Let $A(x)$ be the generating function

From the recurrence, we set

$$a_{n+2}\,x^{n+2} - 2a_{n+1}\,x^{n+2} + a_n\,x^{n+2} = 2^n\,x^{n+2}$$

Summing the equations for all n, we get :

$$(A(x)–2x–1) – 2x(A(x)–1) + x^2A(x) = x^2 / (1–2x)$$

Simplifying the above, we get :

$$A(x) = 1 / (1–2x) \quad \Rightarrow \quad a_n=2^n, \quad n \geqq 0$$

Solution :

(b) Let $A(x)$ and $B(x)$ be the generating functions

From the recurrences, we have :

$$A(x) \quad = 2x\,A(x) - x\,B(x) + 2x/(1-x)$$

$$B(x) - 1 = -x\,A(x) + 2x\,B(x) - x/(1-x)$$

Solving the above, we get :

$$A(x) = \frac{x - 2x^2}{(1-x)^2(1-3x)} = \frac{3x - 1}{4(1-x)^2} + \frac{1}{4(1-3x)}$$

$$= \frac{-3}{4(1-x)} + \frac{1}{2(1-x)^2} + \frac{1}{4(1-3x)}$$

Solution (cont) :

(b) Also, we get :

$$B(x) = \frac{2x^2 - 4x + 1}{(1-x)^2(1-3x)} = \frac{-3x+5}{4(1-x)^2} - \frac{1}{4(1-3x)}$$

$$= \frac{3}{4(1-x)} + \frac{1}{2(1-x)^2} - \frac{1}{4(1-3x)}$$

Thus, we have :

$$a_n = (-3/4) + (1/2)(n+1) + (1/4)3^n, \quad n \geqq 0$$

$$b_n = (3/4) + (1/2)(n+1) - (1/4)3^n, \quad n \geqq 0$$

Question 4

- Let a_n denote the number of n-bit binary strings in which the pattern 111 occurs exactly twice, with the second occurrence at the nth bit.

  Find the generating function for $(a_0, a_1, \ldots)$.

Solution :

Let b_n denote the number of n-bit binary strings in which the pattern 111 occurs for the *first* time at position n

Among all the n-bit binary sequences, there are 2^{n-3} sequences that have 111 as the last three bits

Thus we have :

$$2^{n-3} = b_n + b_{n-1} + b_{n-2} + b_{n-3}\, 2 + b_{n-4}\, 2^1 + \ldots + b_3\, 2^{n-6}$$

Solution (cont) :

Let $b_0 = b_1 = b_2 = 0$

Let $C(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots$

$= 1 + x + x^2 + 2^0x^3 + \ldots + 2^{n-3}x^n + \ldots$

$= 1 + x + x^2 + [x^3 / (1-2x)]$

$$\sum_{n=3}^{\infty} 2^{n-3} x^n = \sum_{n=3}^{\infty} (b_n c_0 + b_{n-1} c_1 + \ldots + b_1 c_{n-1} + b_0 c_n) x^n$$

➔ $C(x) - 1 - x - x^2 = B(x)C(x)$

➔ $B(x) = x^3/(1 - x - x^2 - x^3)$

Solution (cont) :

$$a_n = b_3 b_{n\text{-}3} + b_4 b_{n\text{-}4} + \ldots + b_{n\text{-}4} b_4 + b_{n\text{-}3} b_3$$

$$\sum_{n=6}^{\infty} a_n x^n = \sum_{n=6}^{\infty} (b_0 b_n + b_1 b_{n-1} + \ldots + b_{n-1} b_1 + b_n b_0) x^n$$

➔ $A(x) = B(x)B(x)$

$$= [\ x^3/(1 - x - x^2 - x^3)\ ]^2$$

Question 5

a) Let b_n denote # of n-bit binary strings in which 11011 *first* occurs at the nth bit.

   Find the generating function for $(b_0, b_1, \ldots)$.

b) Let c_n denote # of n-bit binary strings in which 11011 occurs at the nth bit

   Find the generating function for $(c_0, c_1, \ldots)$.

Solution :

(a)

Among all the n-bit binary sequences, there are 2^{n-5} sequences that have 11011 as the last 5 bits

$$2^{n-5} = b_n + b_{n-3} + b_{n-4} + b_{n-5}2^0 + b_{n-6}2^1 + \ldots + b_5 2^{n-10}$$

Solution (cont) :

Let $b_0 = b_1 = b_2 = b_3 = b_4 = 0$

Let $A(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$

$= 1 + x^3 + x^4 + 2^0x^5 + \ldots + 2^{n-5}x^n + \ldots$

$= 1 + x^3 + x^4 + [x^5/(1-2x)]$

$$\sum_{n=5}^{\infty} 2^{n-5} x^n = \sum_{n=5}^{\infty} (b_n a_0 + b_{n-1} a_1 + \ldots + b_1 a_{n-1} + b_0 a_n) x^n$$

➔ $A(x) - 1 - x^3 - x^4 = B(x)A(x)$

➔ $B(x) = x^5/(1 - 2x + x^3 - x^4 - x^5)$

Solution (cont) :

(b)

Among all the n-bit binary sequences, there are 2^{n-5} sequences that have 11011 as the last 5 bits

Thus, we have : $2^{n-5} = c_n + c_{n-3} + c_{n-4}$.

Set $c_0 = 1$, $c_1 = c_2 = c_3 = c_4 = 0$.

Solution :

$$\sum_{n=5}^{\infty} 2^{n-5} x^n = \sum_{n=5}^{\infty} c_n x^n + \sum_{n=5}^{\infty} c_{n-3} x^n + \sum_{n=5}^{\infty} c_{n-4} x^n$$

$$\frac{x^5}{1-2x} = (C(x)-1) + x^3(C(x)-1) + x^4(C(x)-1)$$

$$C(x) = \frac{1-2x+x^3-x^4-x^5}{1-2x+x^3-x^4-2x^5}$$

Question 6

- Let d_n be the number of ways to completely cover a $2 \times n$ rectangle with 2×1 dominoes.

- Express d_n in terms of n

Solution :

$$d_n = d_{n-1} + d_{n-2}, \quad \text{for } n \geqq 2.$$

$$d_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$