CS2335 Special Topics in Discrete Structure

Homework 1

Due: 1:10 pm, October 5, 2009 (before class)

1. Prove the following identity using combinatorial arguments:

$$C(n+1,m) = C(n,m) + C(n-1,m-1) + C(n-2,m-2) + \cdots + C(n-m,0) \quad \text{for } m \le n$$

2. Prove the following identity using combinatorial arguments:

$$n \times C(n-1,r) = (r+1) \times C(n,r+1).$$

3. (a) Using combinatorial arguments, prove that $(2n)!/2^n$ is an integer.
   (b) Prove that $(3n)!/(2^n \times 3^n)$ is an integer. (*Hint:* 2 and 3 are relatively prime.)

4. There is a new fast food restaurant in town. If you have k dollars to spend (k is an integer), their menu has exactly k^2+1 different sets with cost k that you may choose.†

   Four identical quadruples come to this restaurant, and they are going to spend exactly 4 dollars in total. How many different configurations can these quadruples try?

   For instance, one possible configuration is: three of them spend 0 dollars, each getting the set that costs 0 dollars, and the last one spends 4 dollars, choosing one set from the 17 sets that costs 4 dollars. Note that the quadruples are identical, so that we do not care which three of them get the sets costing 0 dollars.

5. Suppose that no three of the diagonals of a convex n-gon meet at the same point inside of the n-gon. Triangles will be formed with the sides of made up of the sides of the n-gon, the diagonals, or segments of the diagonals. How many different triangles are there?

   For instance, when $n = 4$, there will be 8 triangles.

6. (Challenging: No marks) How many permutations of the integers $1, 2, \ldots, n$ are there such that every integer is followed by (but not necessarily immediately followed by) an integer which differs from it by 1?

   For example, with $n = 4$, 1432 is an acceptable permutation but 2431 is not.

7. (Challenging: No marks) Consider a parenthesis sequence with n `(`'s and n `)`'s. Such a sequence is called *valid* if for any $m \le n$, the number of `(`'s in the first m symbols is at least the number of `)`'s in the first m symbols. Otherwise, a sequence is called *invalid*. How many valid sequences are there?

   For instance, `()()` or `(())` are both valid, while `)(()` and `())(` are both invalid.

   Hint: Given an invalid sequence, we are going to obtain a *modified* sequence as follows: (1) Suppose k is the smallest number such that there are more `)` than `(` in the first k symbols. (2) Flip the sequence starting from the $k+1$th symbol. What is so special about the modified sequence? How many are there?

† That means, if you have no money, you still have something to eat!!