CS5371 Theory of Computation

Homework 1
Due: 3:20 pm, October 13, 2006 (before class)

1. Assume that the alphabet is $\{\texttt{0}, \texttt{1}\}$. Give the state diagram of a DFA that recognizes the language $\{w \mid w \text{ ends with } \texttt{00}\}$.

2. Assume that the alphabet is $\{\texttt{0}, \texttt{1}\}$. Give the state diagram of a DFA that recognizes the language $\{w \mid w \text{ contains an equal number of occurrences of the substrings } \texttt{01} \text{ and } \texttt{10}\}$.

3. Prove that the language $\{w^p \mid p \text{ is prime}\}$ is not regular. (You may assume that the number of primes is infinite.)

4. Consider the language $F = \left\{\texttt{a}^i\texttt{b}^j\texttt{c}^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\right\}$.

   (a) Show that F is not regular.

   (b) Show that F acts like a regular language in the pumping lemma. In other words, give a pumping length p and demonstrate that F satisfies the three conditions of the pumping lemma for this value of p.

   (c) Explain why parts (a) and (b) do not contradict the pumping lemma.

5. For languages A and B, let the *perfect shuffle* of A and B be the language

$$\{w \mid w = \texttt{a}_1\texttt{b}_1 \cdots \texttt{a}_k\texttt{b}_k, \text{ where } \texttt{a}_1 \cdots \texttt{a}_k \in A \text{ and } \texttt{b}_1 \cdots \texttt{b}_k \in B, \text{ each } \texttt{a}_i, \texttt{b}_i \in \Sigma\}.$$

   Show that the class of regular languages is closed under perfect shuffle.

6. For languages A and B, let the *shuffle* of A and B be the language

$$\{w \mid w = \texttt{a}_1\texttt{b}_1 \cdots \texttt{a}_k\texttt{b}_k, \text{ where } \texttt{a}_1 \cdots \texttt{a}_k \in A \text{ and } \texttt{b}_1 \cdots \texttt{b}_k \in B, \text{ each } \texttt{a}_i, \texttt{b}_i \in \Sigma^*\}.$$

   Show that the class of regular languages is closed under shuffle.

7. **(Myhill-Nerode Theorem.)** Let L be any language.

   Definition 1. *Let x and y be strings. We say that x and y are* distinguishable by L *if some string z exists whereby exactly one of the strings xz and yz is a member of L; otherwise, for every string z, we have $xz \in L$ whenever $yz \in L$.*

   Definition 2. *Let X be a set of strings. We say that X is* pairwise distinguishable by L *if every two distinct strings in X are distinguishable by L.*

   Definition 3. *Define the* index *of L to be the maximum number of elements in any set of strings that is pairwise distinguishable by L. The index of L may be finite or infinite.*

   (a) Show that if L is recognized by a DFA with k states, L has index at most k.

   (b) Show that, if the index of L is a finite number k, it is recognized by a DFA with k states.

   (c) Conclude that L is regular if and only if it has finite index. Moreover, its index is the size of the smallest DFA recognizing it.

8. (Bonus Question) If A is any language, let $A_{\frac{1}{2}}$ be the set of all first halves of strings in A so that

$$A_{\frac{1}{2}} = \{x \mid \text{ for some } y, |x| = |y| \text{ and } xy \in A\}.$$

Show that, if A is regular, then so is $A_{\frac{1}{2}}$.