CS4311
Design and Analysis of Algorithms

Tutorial for Fun:
Deriving Catalan Number Formula

Catalan Number

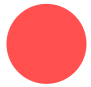
- Let us define the n^{th} Catalan number

  c_n = # binary trees with n internal nodes
  = # binary trees with $n+1$ leaves

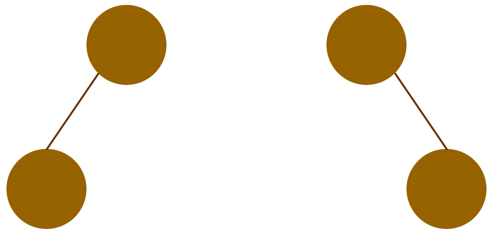
- What is c_0, c_1, c_2, c_3 ?

$c_0 = 1$

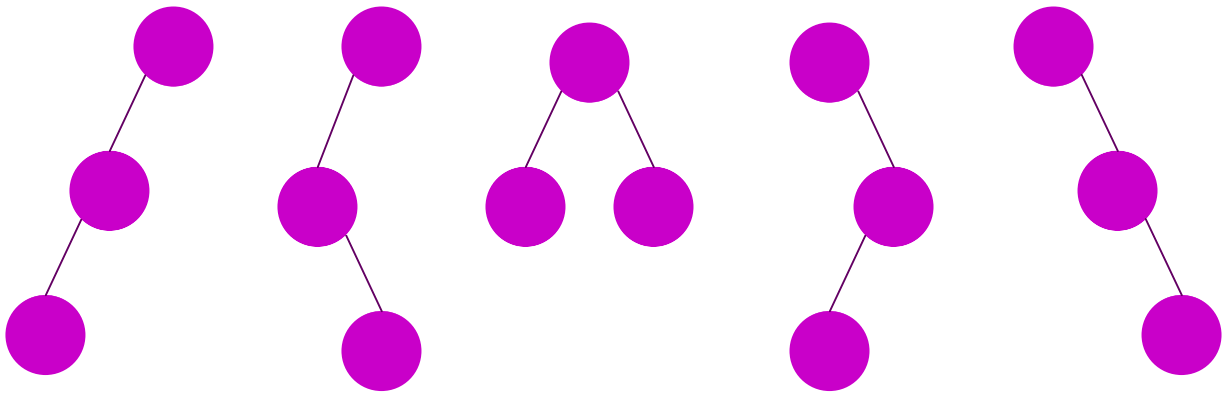
$c_1 = 1$



$c_2 = 2$



$c_3 = 5$



Catalan Number

- Note: an n-node tree can be formed by:
  (i) choosing the k^{th} node to be its root
  (ii) arrange the left tree in any order
  (iii) arrange the right tree in any order
  So, there are $c_{k-1} * c_{n-k}$ choices

→ $c_n = c_0 c_{n-1} + c_1 c_{n-2} + c_2 c_{n-3} + \ldots + c_{n-1} c_0$

$= \sum_{k=1 \text{ to } n} c_{k-1} c_{n-k}$

Generating Function

- Let $S = s_0, s_1, s_2, \ldots$ be a series of numbers we are interested
- Then the function

$$F(x) = \sum s_i x^i = s_0 + s_1 x + s_2 x^2 + s_3 x^3 + \ldots$$

is called a generating function of S

Generating Function

Example 1:

$$F(x) = \sum i\, x^i = x + 2x^2 + 3x^3 + \ldots$$

is the generating function of 0, 1, 2, …

Example 2:

$$F(x) = 1 + 4x + 6x^2 + 4x^3 + x^4$$

is the generating function of

$$\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$$

Closed Form

- Sometimes, generating function can be expressed in the closed form :

Example 1:

$$F(x) = \sum x^i = 1 + x + x^2 + x^3 + \dots$$

has a closed form $1 / (1-x)$

Why? Because $(1-x)(1 + x + x^2 + x^3 + \dots) = 1$

Closed Form

Example 2:

$$F(x) = \sum C(n,i)\, x^i$$
$$= 1 + nx + C(n,2)x^2 + \ldots nx^{n-1} + x^n$$

has a closed form $(1+x)^n$

Example 3: How about the closed form of

$$F(x) = \sum i\, x^i = x + 2x^2 + 3x^3 + \ldots \; ?$$

Closed Form

- Generating function is very useful in
  (I) solving combinatorial problems, and
  (II) solving recurrences

- Usually, the closed form is important because it can simplify the notation a lot!

- We will see how generating function is used to get Catalan number formula

Catalan Number

So, we have:

$$c_0 = 1$$

$$c_1 x = c_0 c_0 x$$

$$\vdots$$

$$c_{n-1} x^{n-1} = \sum_{k=1 \text{ to } n-1} c_{k-1} c_{n-k} x^{n-1}$$

$$c_n x^n = \sum_{k=1 \text{ to } n} c_{k-1} c_{n-k} x^n$$

$$\vdots$$

Generating Function

Let $F(x)$ = generating function of Catalan #

$= c_0 + c_1x + \ldots + c_nx^n + \ldots$

= sum of LHS

= sum of RHS

$= 1 + x\,[\, c_0c_0 + (c_0c_1 + c_1c_0)x + \ldots$
$\quad + (c_0c_{n-1} + \ldots + c_{n-1}c_0)x^{n-1} + \ldots \,]$

$= 1 + x\,(F(x))^2$

Closed Form

Thus,

$$F(x) = 1 + x\,(F(x))^2$$

Or,

$$x\,(F(x))^2 - F(x) + 1 = 0$$

Hence, we get a closed form of $F(x)$:

$$F(x) = (\,1 \pm \sqrt{1 - 4x}\,)\,/\,(2x)$$

$$= (\,1 - \sqrt{1 - 4x}\,)\,/\,(2x)$$ (why?)

Closer Look on Closed Form

We claim that:

$(1-4x)^{1/2} = 1 - 2x - \ldots - C(2n,n)/(2n-1)\ \ x^n - \ldots$

If the claim is correct, then:

$F(x) = 1 - ((1-4x)^{1/2}) / (2x)$

$= 1 + \ldots + C(2n,n)/(2(2n-1))\ x^{n-1} + \ldots$

so that

$c_n = C(2n+2,n+1) / (2(2n+1))$

Getting the Catalan Number

➔ $c_n = C(2n+2,n+1) / (2(2n+1))$

$= (2n+2)! / ((n+1)! (n+1)! 2(2n+1))$

$= (2n+2)! / (n! (n+1)! (2n+2)(2n+1))$

$= (2n)! / (n! (n+1)!)$

$= (2n)! / (n! n! (n+1))$

$= C(2n,n) / (n+1)$

Proof of the Claim

Let $C(\frac{1}{2},k) = \frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-k+1) / k!$

By Binomial Expansion, (or Taylor)

$(1 - 4x)^{1/2}$

$= 1 + \frac{1}{2}(-4x) + \ldots + C(\frac{1}{2},n)(-4x)^n + \ldots$

$= 1 - 2x - \ldots - 4^n \frac{1}{2}(1-\frac{1}{2})(2-\frac{1}{2})\ldots(n-1-\frac{1}{2})x^n/n!$

$- \ldots$

Simplifying Terms

$$4^n \tfrac{1}{2}(1-\tfrac{1}{2})(2-\tfrac{1}{2})\ldots(n-1-\tfrac{1}{2}) \;/\; n!$$

$$= \;2^n (1)(1)(3)(5)\ldots(2n-3) \;/\; n!$$

$$= 2^n\, n!\, (1)(3)(5)\ldots(2n-3)(2n-1)/(n!\; n!(2n-1))$$

$$= (2)(4)(6)\ldots(2n)(1)(3)(5)\ldots(2n-1)/(n!\; n!(2n-1))$$

$$= (2n)! \;/\; (n!\; n!\; (2n-1))$$