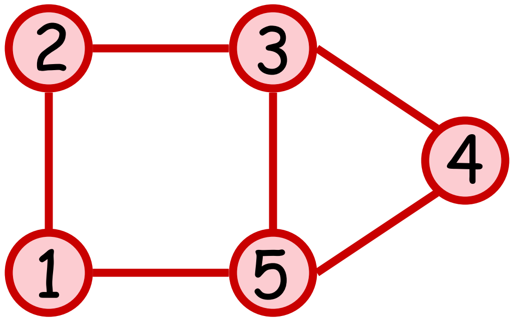
CS4311
Design and Analysis of Algorithms

Lecture 22:
Elementary Graph Algorithms I

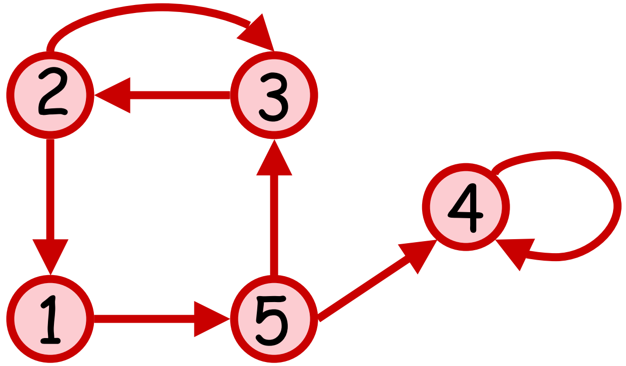
About this lecture

- Representation of Graph
  - Adjacency List, Adjacency Matrix
- Breadth First Search

Graph



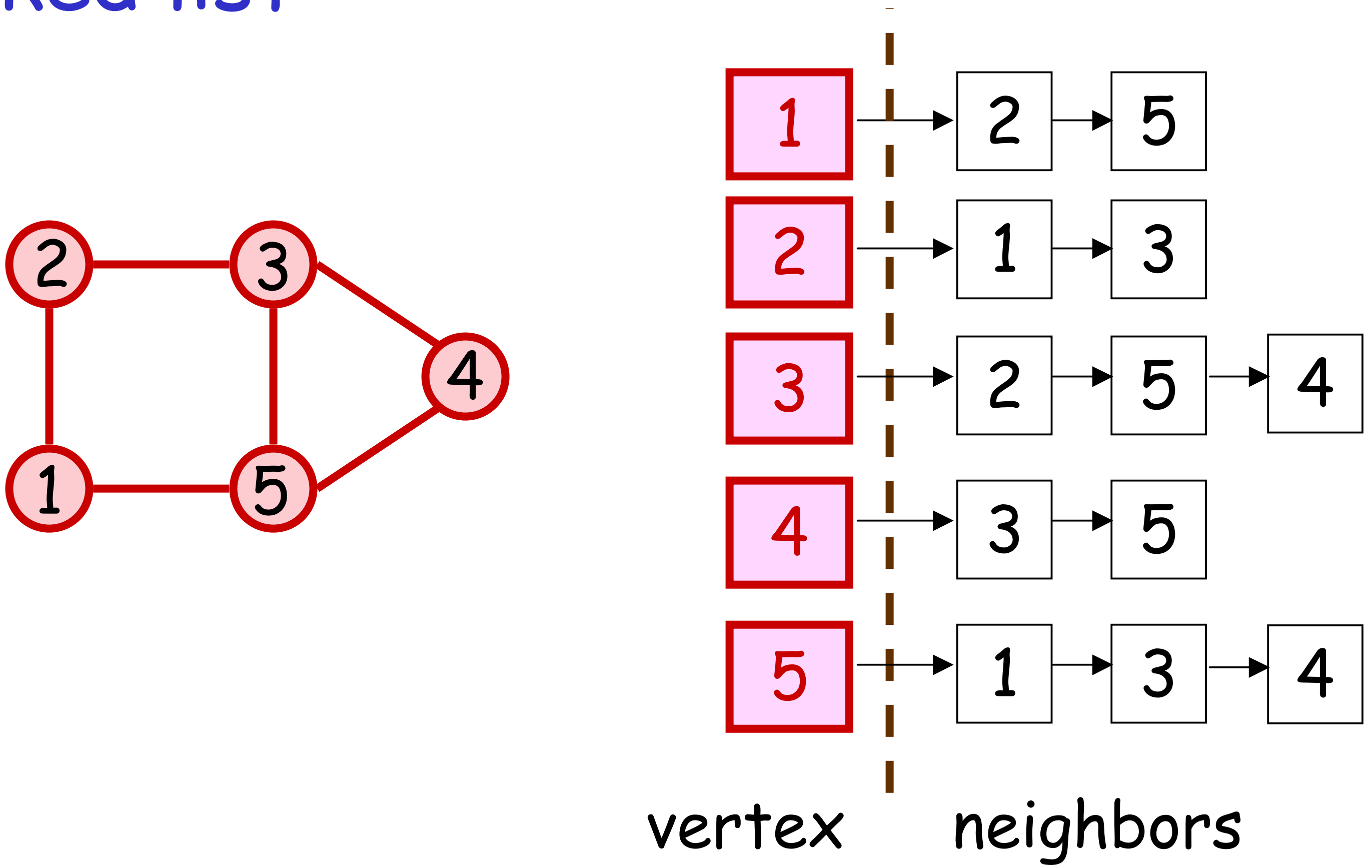
undirected



directed

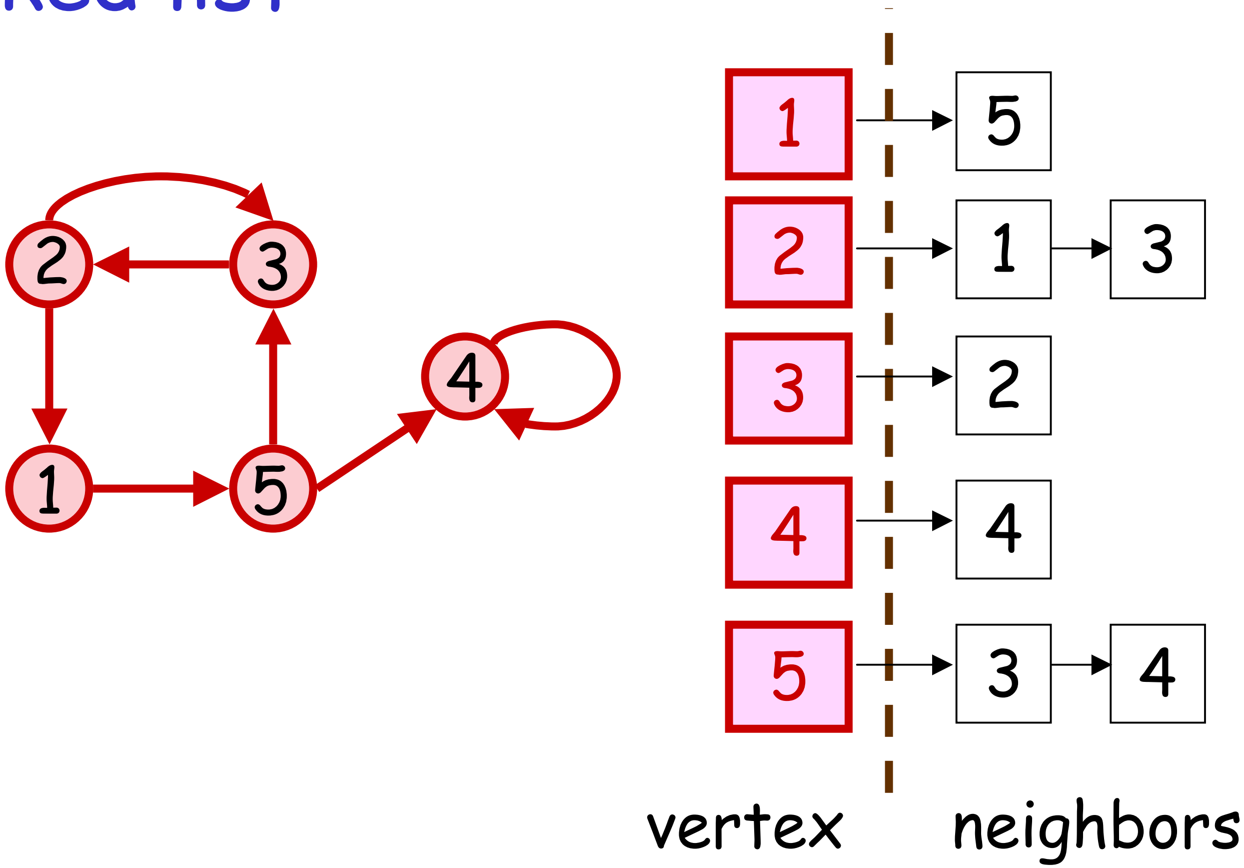
Adjacency List (1)

- For each vertex u, store its neighbors in a linked list



Adjacency List (2)

- For each vertex u, store its neighbors in a linked list



Adjacency List (3)

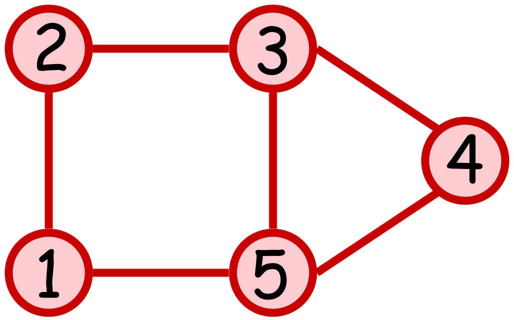
- Let $G = (V, E)$ be an input graph
- Using Adjacency List representation :
  - Space : $O(|V| + |E|)$

    ➔ Excellent when $|E|$ is small
  - Easy to list all neighbors of a vertex
  - Takes $O(|V|)$ time to check if a vertex u is a neighbor of a vertex v
- can also represent weighted graph

Adjacency Matrix (1)

- Use a $|V| \times |V|$ matrix A such that

$$A(u,v) = 1 \quad \text{if} \quad (u,v) \text{ is an edge}$$
$$A(u,v) = 0 \quad \text{otherwise}$$

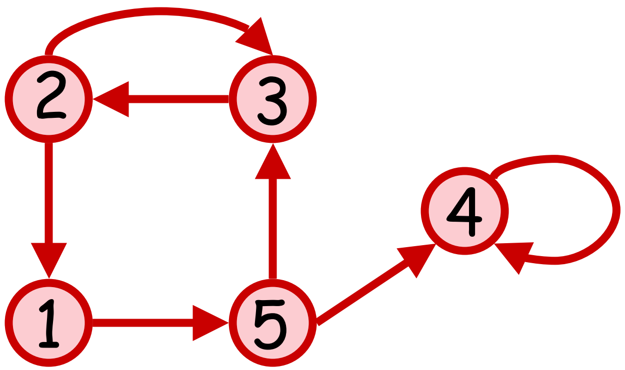


	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	0	0
3	0	1	0	1	1
4	0	0	1	0	1
5	1	0	1	1	0

Adjacency Matrix (2)

- Use a $|V| \times |V|$ matrix A such that

$$A(u,v) = 1 \quad \text{if} \quad (u,v) \text{ is an edge}$$
$$A(u,v) = 0 \quad \text{otherwise}$$



	1	2	3	4	5
1	0	0	0	0	1
2	1	0	1	0	0
3	0	1	0	0	0
4	0	0	0	1	0
5	0	0	1	1	0

Adjacency Matrix (3)

- Let $G = (V, E)$ be an input graph
- Using Adjacency Matrix representation :
  - Space : $O(|V|^2)$

    ➔ Bad when $|E|$ is small
  - $O(1)$ time to check if a vertex u is a neighbor of a vertex v
  - $\Theta(|V|)$ time to list all neighbors
- can also represent weighted graph

Transpose of a Matrix

- Let A be an $n \times m$ matrix

Definition:

The transpose of A, denoted by A^T, is an $m \times n$ matrix such that

$$A^T(u,v) = A\,(v,u) \quad \text{for every } u, v$$

➔ If A is an adjacency matrix of an undirected graph, then $A = A^T$

Breadth First Search (BFS)

- A simple algorithm to find all vertices reachable from a particular vertex s
  - s is called source vertex
- Idea: Explore vertices in rounds
  - At Round k, visit all vertices whose shortest distance (#edges) from s is k-1
  - Also, discover all vertices whose shortest distance from s is k

The BFS Algorithm

1. Mark s as discovered in Round 0
2. For Round k = 1, 2, 3, …,

   For (each u discovered in Round k-1)

   { Mark u as visited ;

   Visit each neighbor v of u ;

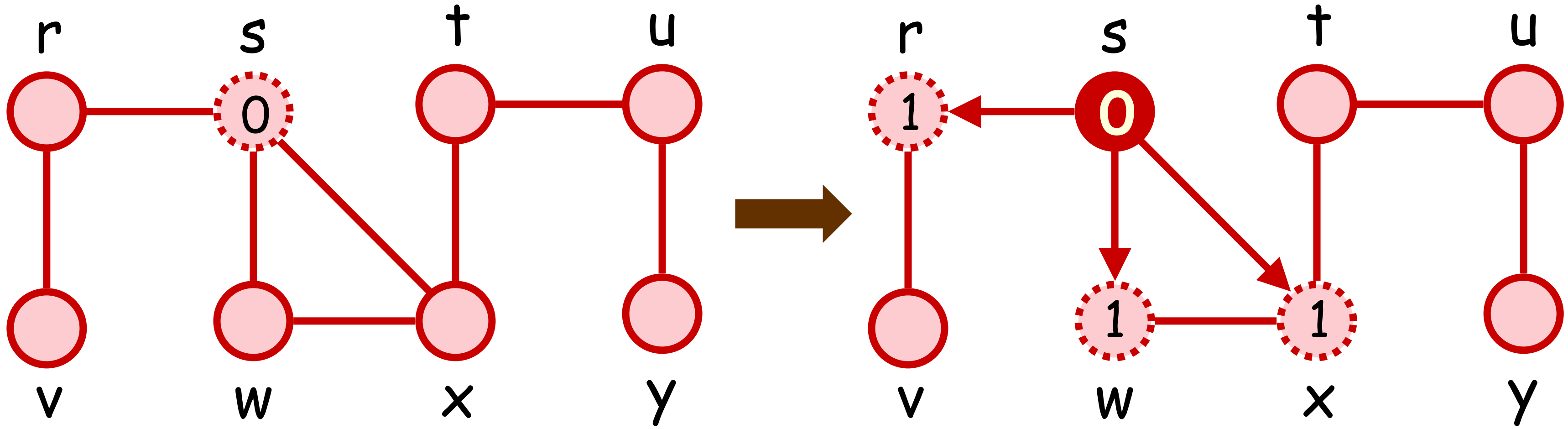
   If (v not visited and not discovered)

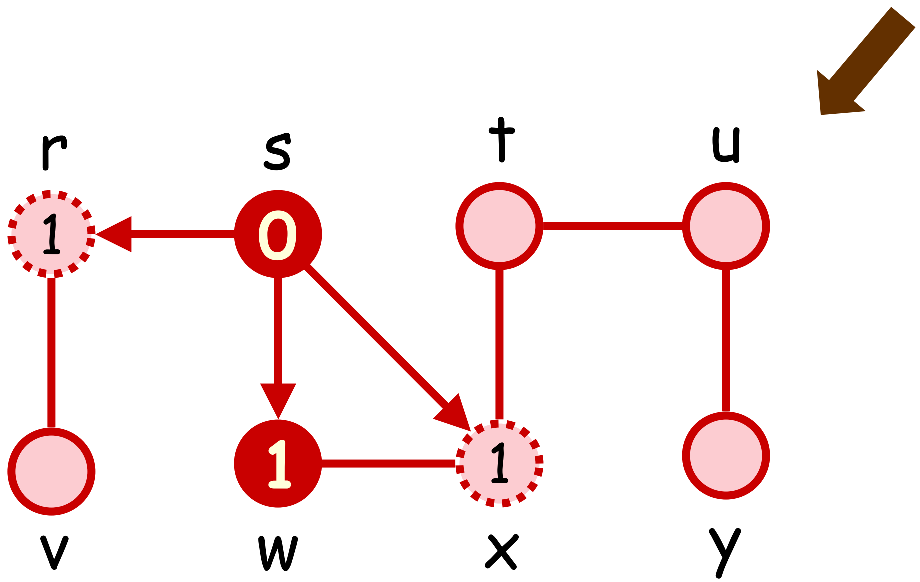
   Mark v as discovered in Round k ;

   }

Stop if no vertices were discovered in Round k-1

Example (s = source)



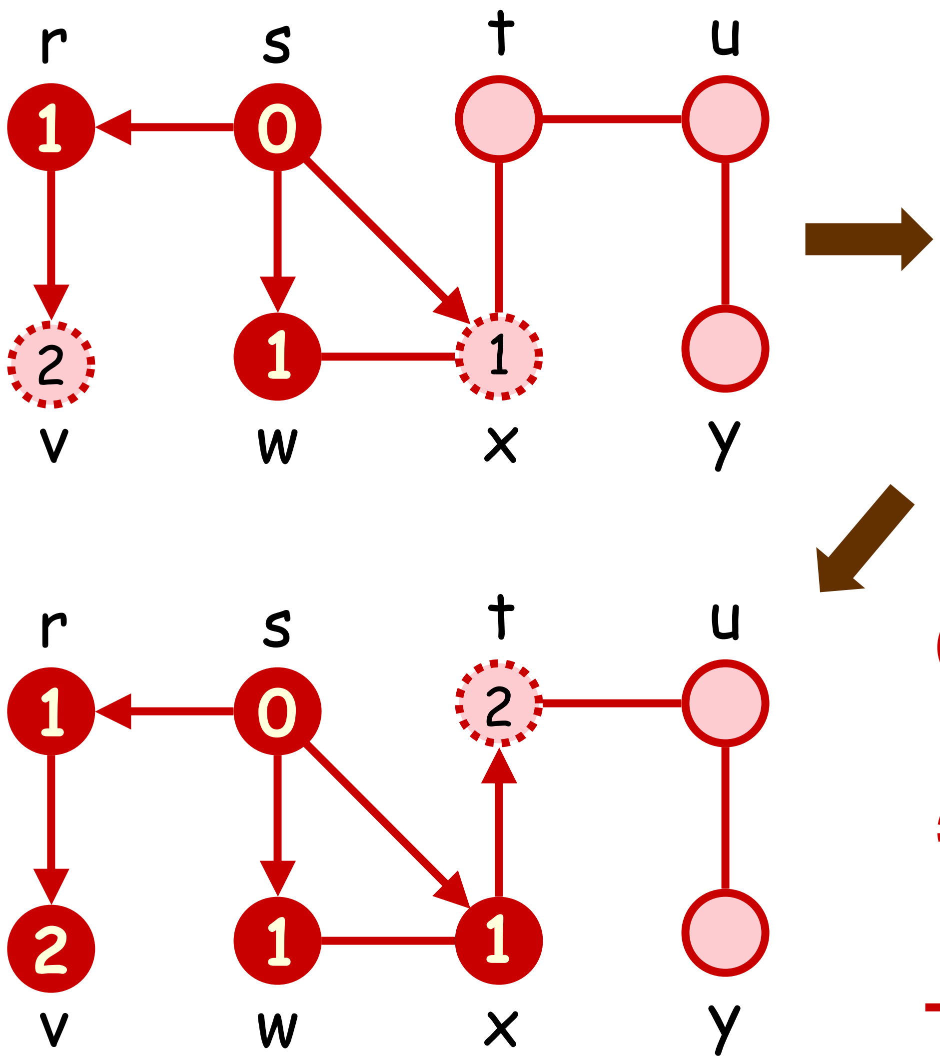


? visited
(? = discover time)

? discovered
(? = discover time)

→ direction of edge when new node is discovered

Example (s = source)

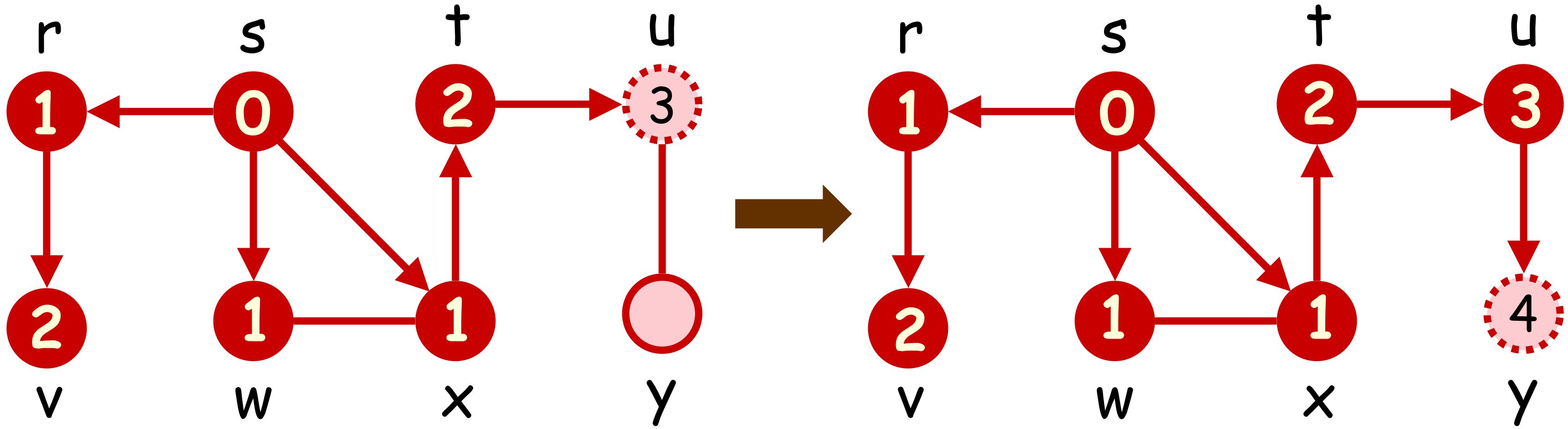


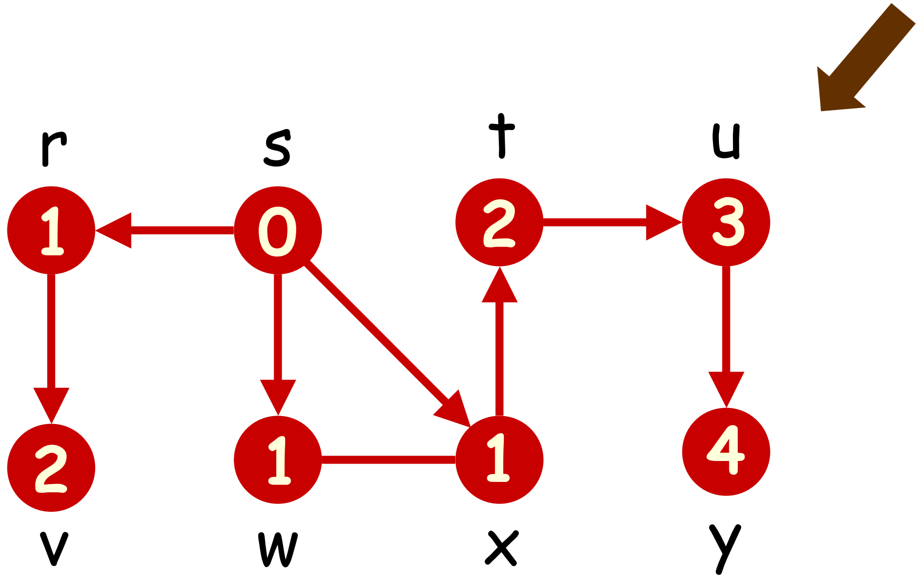
visited
(? = discover time)

discovered
(? = discover time)

direction of edge when new node is discovered

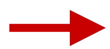
Example (s = source)





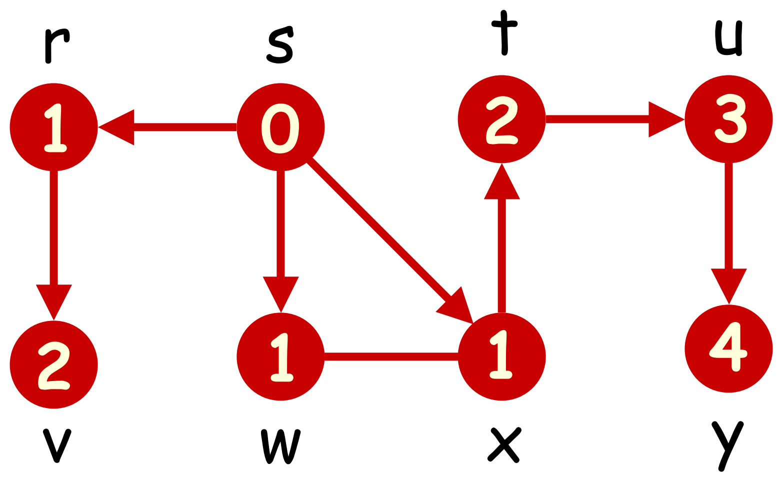
visited
(? = discover time)

discovered
(? = discover time)

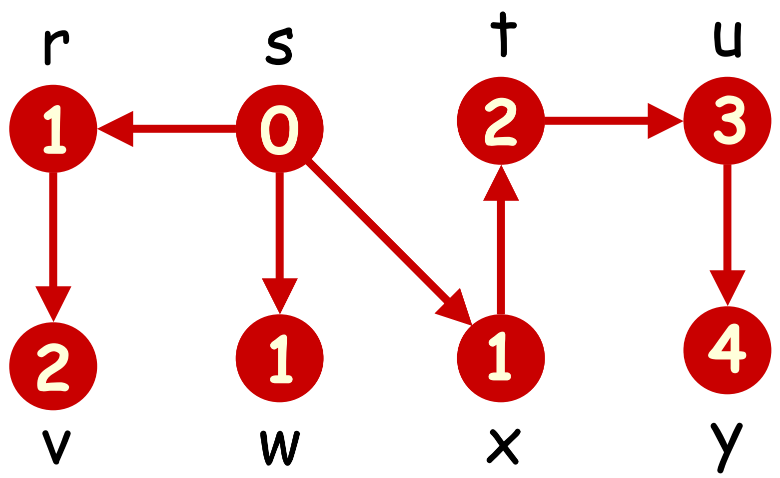


direction of edge when new node is discovered

Example (s = source)



Done when no new node is discovered



The directed edges form a tree that contains all nodes reachable from s

Called BFS tree of s

Correctness

- The correctness of BFS follows from the following theorem :

Theorem: A vertex v is discovered in Round k if and only if shortest distance of v from source s is k

Proof: By induction

Performance

- BFS algorithm is easily done if we use
  - an $O(|V|)$-size array to store discovered/visited information
  - a separate list for each round to store the vertices discovered in that round
- Since no vertex is discovered twice, and each edge is visited at most twice (why?)

→ Total time: $O(|V|+|E|)$

→ Total space: $O(|V|+|E|)$

Performance (2)

- Instead of using a separate list for each round, we can use a common queue
  - When a vertex is discovered, we put it at the end of the queue
  - To pick a vertex to visit in Step 2, we pick the one at the front of the queue
  - Done when no vertex is in the queue

➔ No improvement in time/space …

➔ But algorithm is simplified

Question: Can you prove the correctness of using queue?