CS4311
Design and Analysis of Algorithms

Tutorial for Fun:
Deriving Catalan Number Formula
Generating Function

• Let $S = s_0, s_1, s_2, \ldots$ be a series of numbers we are interested in.

• Then the function

$$F(x) = \sum s_i x^i = s_0 + s_1x + s_2x^2 + s_3x^3 + \ldots$$

is called a generating function of $S$. 

Generating Function

Example 1:

\[ F(x) = \sum_i i x^i = x + 2x^2 + 3x^3 + \ldots \]

is the generating function of 0, 1, 2, ...

Example 2:

\[ F(x) = 1 + 4x + 6x^2 + 4x^3 + x^4 \]

is the generating function of

\[
\begin{pmatrix}
4 \\
0
\end{pmatrix}, \begin{pmatrix}
4 \\
1
\end{pmatrix}, \begin{pmatrix}
4 \\
2
\end{pmatrix}, \begin{pmatrix}
4 \\
3
\end{pmatrix}, \begin{pmatrix}
4 \\
4
\end{pmatrix}
\]
Closed Form

• Sometimes, generating function can be expressed in the closed form:

Example 1:

\[ F(x) = \sum x^i = 1 + x + x^2 + x^3 + \ldots \]

has a closed form \( 1 / (1-x) \)

Why? Because \((1-x)(1 + x + x^2 + x^3 + \ldots) = 1\)
Closed Form

Example 2:

\[ F(x) = \sum C(n,i) x^i \]

\[ = 1 + nx + C(n,2)x^2 + \ldots nx^{n-1} + x^n \]

has a closed form \((1+x)^n\)

Example 3: How about the closed form of

\[ F(x) = \sum i x^i = x + 2x^2 + 3x^3 + \ldots \]?
Closed Form

• Generating function is very useful in (I) solving combinatorial problems, and (II) solving recurrences

• Usually, the closed form is important because it can simplify the notation a lot!

• We will see how generating function is used to get Catalan number formula
Catalan Number

• Let us define the $n^{th}$ Catalan number

$$c_n = \# \text{ binary trees with } n \text{ internal nodes}$$

$$= \# \text{ binary trees with } n+1 \text{ leaves}$$

• What is $c_0$, $c_1$, $c_2$, $c_3$?
$c_0 = 1$

$c_1 = 1$

$c_2 = 2$

$c_3 = 5$
Catalan Number

• Note: an \( n \)-node tree can be formed by:
  (i) choosing the \( k^{th} \) node to be its root
  (ii) arrange the left tree in any order
  (iii) arrange the right tree in any order

So, there are \( c_{k-1} \times c_{n-k} \) choices

\[
\Rightarrow c_n = c_0c_{n-1} + c_1c_{n-2} + c_2c_{n-3} + \ldots + c_{n-1}c_0
\]

\[
= \sum_{k=1 \text{ to } n} c_{k-1}c_{n-k}
\]
Catalan Number

So, we have:

\[ c_0 = 1 \]
\[ c_1 = c_0c_0 \times \]
\[ \vdots \]
\[ c_{n-1} \times^{n-1} = \sum_{k=1 \text{ to } n-1} c_{k-1}c_{n-k} \times^{n-1} \]
\[ c_n \times^{n} = \sum_{k=1 \text{ to } n} c_{k-1}c_{n-k} \times^{n} \]
\[ \vdots \]
Catalan Number

Let $F(x) =$ generating function of Catalan #

$$= c_0 + c_1 x + \ldots + c_n x^n + \ldots$$

$$= \text{sum of LHS}$$

However,

sum of RHS

$$= 1 + x \left[ c_0 c_0 + (c_0 c_1 + c_1 c_0) x + \ldots + (c_0 c_{n-1} + \ldots + c_{n-1} c_0) x^{n-1} + \ldots \right]$$

$$= 1 + x (F(x))^2$$
Catalan Number

Thus,

\[ F(x) = 1 + x (F(x))^2 \]

Or,

\[ x (F(x))^2 - F(x) + 1 = 0 \]

Hence, we get a closed form of \( F(x) \):

\[ F(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x} \]

\[ = \frac{1 - \sqrt{1 - 4x}}{2x} \quad \text{(why?)} \]
Catalan Number

Let $C\left(\frac{1}{2}, k\right) = \frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) \ldots \left(\frac{1}{2} - k + 1\right) / k!$

Then, by binomial expansion, (or Taylor)

$(1 - 4x)^{1/2}$

$= 1 + \frac{1}{2}(-4x) + \ldots + C\left(\frac{1}{2}, n\right)(-4x)^n + \ldots$

$= 1 - 2x - \ldots - 4^n \frac{1}{2} \left(1 - \frac{1}{2}\right) \left(2 - \frac{1}{2}\right) \ldots \left(n - 1 - \frac{1}{2}\right)x^n / n!$

$- \ldots$
Simplifying Terms

We claim that:

\[ 4^n \left(1 - \frac{1}{2}\right)(2 - \frac{1}{2})\ldots(n-1-\frac{1}{2}) / n! = \binom{2n}{n} / (2n-1) \]

\[ (1-4x)^{1/2} = 1 - 2x - \ldots - \binom{2n}{n}x^n/(2n-1) - \ldots \]

\[ F(x) = 1 - \left( (1-4x)^{1/2} \right) / (2x) = 1 + \ldots + \binom{2n}{n}x^{n-1}/(2(2n-1)) + \ldots \]
Simplifying Terms

Proof of claim:

\[
4^n \frac{\frac{1}{2}(1-\frac{1}{2})(2-\frac{1}{2})...(n-1-\frac{1}{2})}{n!} = 2^n \frac{(1)(1)(3)(5)...(2n-3)}{n!}
\]

\[
= 2^n \frac{n! (1)(3)(5)...(2n-3)(2n-1)/(n! n!(2n-1))}{n! n!(2n-1)}
\]

\[
= (2)(4)(6)...(2n)(1)(3)(5)...(2n-1)/(n! n!(2n-1))
\]

\[
= (2n)! / (n! n! (2n-1))
\]
Catalan Number

Recall: \( n^{\text{th}} \) Catalan number \( c_n \)

\[
= \text{coefficient of } x^n \text{ in } F(x)
\]

\[
\Rightarrow c_n = \binom{2n+2}{n+1} / (2^{2n+1})
\]

\[
= (2n+2)! / ( (n+1)! (n+1)! 2^{2n+1} )
\]

\[
= (2n+2)! / ( n! (n+1)! (2n+2)(2n+1) )
\]

\[
= (2n)! / ( n! (n+1)! )
\]

\[
= (2n)! / ( n! n! (n+1) )
\]

\[
= \binom{2n}{n} / (n+1)
\]