CS4311
Design and Analysis of Algorithms

Lecture for Fun: Ackermann Function

About this lecture

- Ackermann Function and its inverse
- Knuth's up-arrow notation

Ackermann Function

- The Ackermann function is defined recursively as follows:
  - For any non-negative integer m and n,

$$A(0,n) = n + 1$$

$$A(m,0) = A(m-1,1)$$

$$A(m,n) = A(m-1, A(m, n-1))$$

- To simplify notations a bit, let us use

$A_m(n)$ to denote $A(m,n)$

Ackermann Function

So, $A_0(n) = n + 1$

$A_m(0) = A_{m-1}(1)$

$$
\begin{aligned}
A_m(n) &= A_{m-1}(A_m(n-1)) \\
&= A_{m-1}(A_{m-1}(A_m(n-2))) \\
&= A_{m-1}(A_{m-1}(A_{m-1}(A_m(n-3)))) \\
&= \ldots \\
&= A_{m-1}(A_{m-1}(\ldots A_{m-1}(A_m(0))\ldots) \\
&= \underbrace{A_{m-1}(A_{m-1}(\ldots A_{m-1}(A_{m-1}}_{n+1 \text{ iterations of } A_{m-1}}(1))\ldots)
\end{aligned}
$$

Ackermann Function

- Let us begin with finding:

$$A_1(1),\ A_2(1),\ A_3(1)$$

➔ $A_1(1) = 2, \quad A_2(1) = 3, \quad A_3(1) = 13$

➔ To get $A_4(1)$ will take a long sequence of computations already

- In fact, $A_m(n)$ has a very beautiful closed form, using Knuth's up-arrow notation

Knuth's Up-Arrow Notation

- Knuth observed that

$$a\ b = \underbrace{a + a + \ldots + a}_{b \text{ copies of } a}$$

and

$$a^b = \underbrace{a \times a \times \ldots \times a}_{b \text{ copies of } a}$$

Knuth's Up-Arrow Notation

- So, he invented a notation as follows:

$$a \uparrow b = \underbrace{a \times a \times \ldots \times a}_{\text{b copies of a}}$$

- Also,

$$a \uparrow\uparrow b = \underbrace{a \uparrow a \uparrow \ldots \uparrow a}_{\text{b copies of a}}$$

$$= a \uparrow (a \uparrow \ldots (a \uparrow a)\ldots) = \underbrace{a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}}_{\text{b copies}}$$

Knuth's Up-Arrow Notation

- In general,

$$a \uparrow^{m} b = a \overbrace{\uparrow\uparrow\ldots\uparrow}^{m \text{ copies of } \uparrow} b$$

$$= \underbrace{a \uparrow^{m-1} a \uparrow^{m-1} \ldots \uparrow^{m-1} a}_{b \text{ copies of } a}$$

$$= a \uparrow^{m-1} (a \uparrow^{m-1} \ldots (a \uparrow^{m-1} a)\ldots)$$

Knuth's Up-Arrow Notation

- Example:

$$2 \uparrow 2 \quad = \quad 2 \times 2 \quad = 4$$

$$2 \uparrow\uparrow 2 \quad = \quad 2 \uparrow 2 \quad = 4$$

$$2 \uparrow\uparrow\uparrow 2 \quad = \quad 2 \uparrow\uparrow 2 \quad = 4$$

$$2 \uparrow^{m} 2 \quad = \quad 2 \uparrow^{m-1} 2 \quad = 4$$

Knuth's Up-Arrow Notation

- Example:

$$2 \uparrow 3 = 2 \times 2 \times 2 = 8$$

$$2 \uparrow\uparrow 3 = 2 \uparrow 2 \uparrow 2 = 2 \uparrow 4$$

$$= 2 \times 2 \times 2 \times 2 = 16$$

$$2 \uparrow\uparrow\uparrow 3 = 2 \uparrow\uparrow 2 \uparrow\uparrow 2 = 2 \uparrow\uparrow 4$$

$$= 2 \uparrow 2 \uparrow 2 \uparrow 2$$

$$= 2 \uparrow 2 \uparrow 4 = 2 \uparrow 16$$

$$= 65536$$

Knuth's Up-Arrow Notation

- Example:

$$3 \uparrow 2 = 3 \times 3 = 9$$

$$3 \uparrow\uparrow 2 = 3^3 = 27$$

$$3 \uparrow\uparrow\uparrow 2 = 3 \uparrow\uparrow 3$$

$$= 3^{3^3}$$

$$= 3^{27}$$

$$= 7625597484987$$

Knuth's Up-Arrow Notation

- Example:

$$3 \uparrow 3 = 3^3 = 27$$

$$3 \uparrow\uparrow 3 = 7625597484987$$

$$\begin{aligned} 3 \uparrow\uparrow\uparrow 3 &= 3 \uparrow\uparrow 3 \uparrow\uparrow 3 \\ &= 3 \uparrow\uparrow 7625597484987 \\ &= \underbrace{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}_{7625597484987 \text{ copies}} \end{aligned}$$

Closed Form of $A_m(n)$

- $A_0(n) = n + 1$
- $A_1(n) = n + 2 \quad = \quad 2 + (n+3) - 3$
- $A_2(n) = 2n + 3 \quad = \quad 2 \times (n+3) - 3$
- $A_3(n) = 2^{(n+3)} - 3 \quad = \quad 2 \uparrow (n+3) - 3$
- $A_4(n) \quad = \quad 2 \uparrow\uparrow (n+3) - 3$
- $A_5(n) \quad = \quad 2 \uparrow\uparrow\uparrow (n+3) - 3$
- $A_m(n) \quad = \quad 2 \uparrow^{m-2} (n+3) - 3$

How to prove? (By induction)

Inverse Ackermann Function

- The Inverse Ackermann function is defined as follows:
  - For any non-negative integer m and n,

$$\alpha(m,n) = \min\{\, k \mid A(k, \lfloor m/n \rfloor) \geq \log n \,\}$$

  - Usually, we define:

$$\begin{aligned} \alpha(n) &= \alpha(n,n) \\ &= \min\{\, k \mid A(k, 1) \geq \log n \,\} \end{aligned}$$

Inverse Ackermann Function

Because $A(0,1) = 2$, $A(1,1) = 3$, $A(2,1) = 5$,

$A(3,1) = 13$, $A(4,1) = 65533$

➔ the values of $\alpha(n)$ is as follows:

- When $n = 1$ to 4: $\alpha(n) = 0$
- When $n = 5$ to 8 : $\alpha(n) = 1$
- When $n = 9$ to 32: $\alpha(n) = 2$
- When $n = 33$ to 2^{13}: $\alpha(n) = 3$
- When $n = 2^{13}+1$ to 2^{65533}: $\alpha(n) = 4$

Inverse Ackermann Function

➔ Since $2^{65533} \gg 10^{80}$,

for all n that we concern,

$$\alpha(n) \leq 4$$

similarly, for all m, n that we concern,

if $m \geq n$, $\alpha(m,n) \leq 4$

if $m < n$, $\alpha(m,n) \leq 5$