CS4311
Design and Analysis of Algorithms

Lecture 9: Dynamic Programming I

About this lecture

- Divide-and-conquer strategy allows us to solve a big problem by handling only smaller sub-problems
- Some problems may be solved using a stronger strategy: dynamic programming

- We will see some examples today

Assembly Line Scheduling

- You are the **boss** of a company which assembles Gundam models to customers

Assembly Line Scheduling

- Normally, to assemble a Gundam model, there are n sequential steps

Step 0: getting model

Step 1: assembling body

Step 2: assembling legs

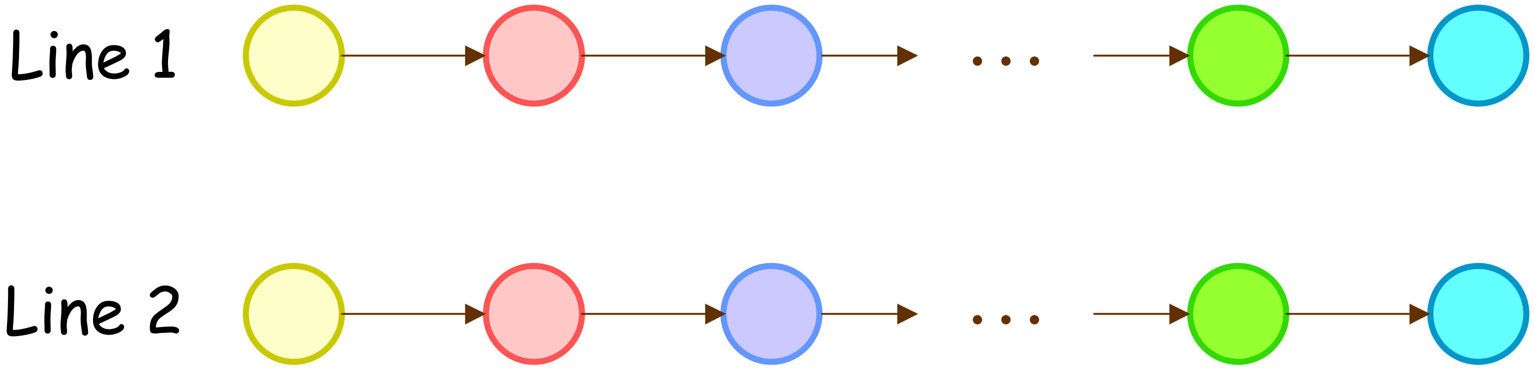
…

Step n-1: polishing

Step n: packaging

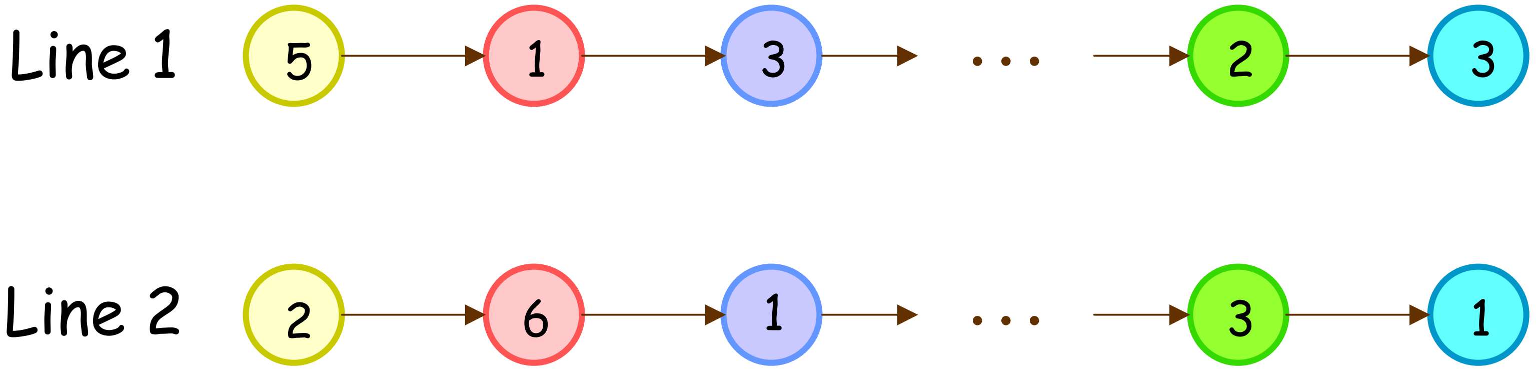
Assembly Line Scheduling

- To improve efficiency, there are two separate assembly lines:



Assembly Line Scheduling

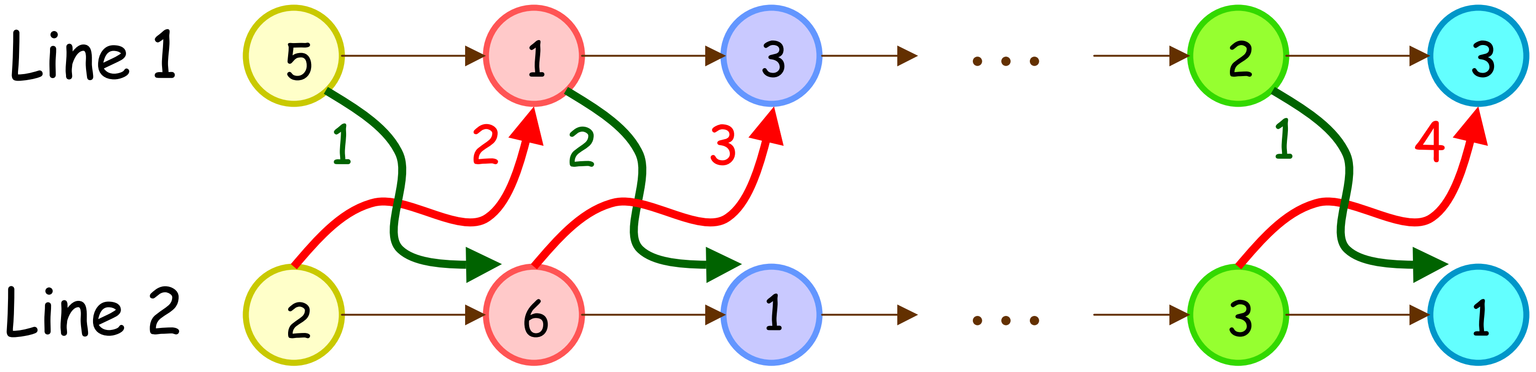
- Since different lines hire different people, processing speed is not the same:



E.g., Line 1 may need 34 mins, and Line 2 may need 38 mins

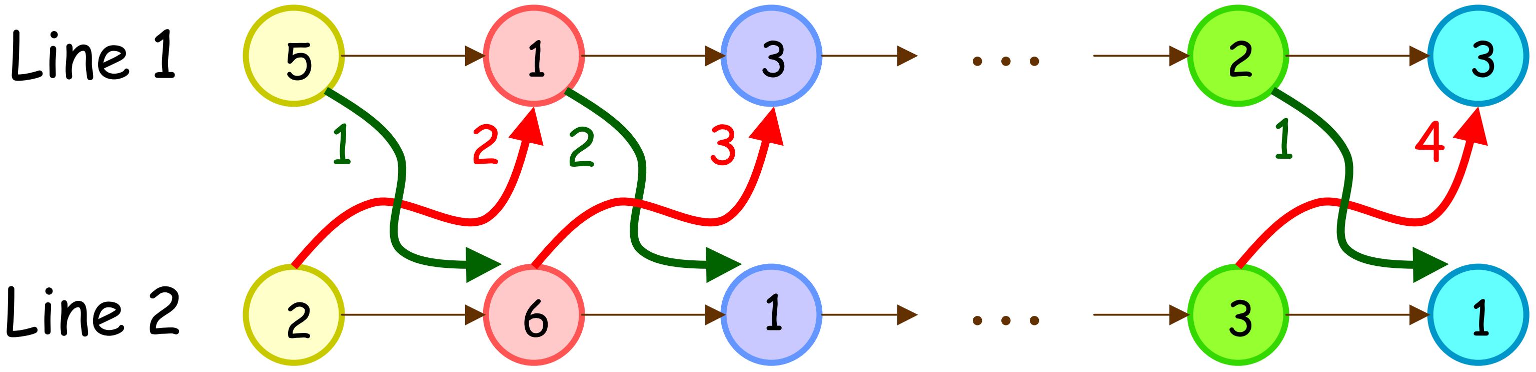
Assembly Line Scheduling

- With some transportation cost, after a step in a line, we can process the model in the other line during the next step



Assembly Line Scheduling

- When there is an urgent request, we may finish faster if we can make use of both lines + transportation in between



E.g., Process Step 0 at Line 2, then process Step 1 at Line 1,
➔ better than process both steps in Line 1

Assembly Line Scheduling

Question: How to compute the fastest assembly time?

Let $p_{1,k}$ = Step k's processing time in Line 1

$p_{2,k}$ = Step k's processing time in Line 2

$t_{1,k}$ = transportation cost from Step k in Line 1 (to Step k+1 in Line 2)

$t_{2,k}$ = transportation cost from Step k in Line 2 (to Step k+1 in Line 1)

Assembly Line Scheduling

Let $f_{1,j}$ = fastest time to finish Steps 0 to j, ending at Line 1

$f_{2,j}$ = fastest time to finish Steps 0 to j, ending at Line 2

So, we have:

$f_{1,0} = p_{1,0}$, $f_{2,0} = p_{2,0}$

fastest time = $\min \{ f_{1,n}, f_{2,n} \}$

Assembly Line Scheduling

Lemma: For any $j > 0$,

$$f_{1,j} = \min \{ f_{1,j-1} + p_{1,j},\quad f_{2,j-1} + t_{2,j-1} + p_{1,j} \}$$

$$f_{2,j} = \min \{ f_{2,j-1} + p_{2,j},\quad f_{1,j-1} + t_{1,j-1} + p_{2,j} \}$$

Proof: By contradiction !

Here, optimal solution to a problem (e.g., $f_{1,j}$) is based on optimal solution to subproblems (e.g., $f_{1,j-1}$ and $f_{2,j-1}$)

→ optimal substructure property

Assembly Line Scheduling

Define a function Compute_F(i,j) as follows:

```
Compute_F(i, j)   /* Finding $f_{i,j}$ */
    if (j == 0) return $p_{i,0}$;
    g = Compute_F(i,j-1) + $p_{i,j}$ ;
    h = Compute_F(3-i,j-1) + $t_{3-i,j-1}$ + $p_{i,j}$ ;
    return min { g, h } ;
```

Calling Compute_F(1,n) and Compute_F(2,n) gives the fastest assembly time

Assembly Line Scheduling

Question: What is the running time of Compute_F(i,n)?

Let $T(n)$ denote its running time

So, $T(n) = 2T(n-1) + \Theta(1)$

➔ By Recursion-Tree Method,

$$T(n) = \Theta(2^n)$$

Assembly Line Scheduling

To improve the running time, observe that:

To Compute_F(1,j) and Compute_F(2,j), both requires the SAME subproblems: Compute_F(1,j-1) and Compute_F(2,j-1)

So, in our recursive algorithm, there are many repeating subproblems which create redundant computations !

Question: Can we avoid it ?

Bottom-Up Approach (Method I)

- In the assembly problem, we notice that
  - (i) all $f_{i,j}$ are eventually computed at least once, and
  - (ii) $f_{i,j}$ depends only on $f_{i,k}$ with $k < j$
- By (i), let us create a 2D table F to store all $f_{i,j}$ values once they are computed
- By (ii), let us compute $f_{i,j}$ from $j = 0$ to n

Bottom-Up Approach (Method I)

```
BottomUp_F( ) /* Finding fastest time */
1. F[1,0] = $p_{i,0}$ ,   F[2,0] = $p_{2,0}$ ;
2. for (j = 1,2,..., n) {
       Compute F[1,j] and F[2,j] ;
       // Based on F[1,j-1] and F[2,j-1]
   }
3. return min { F[1,n] , F[2,n] } ;
```

Running Time = $\Theta(n)$

Memoization (Method II)

- Similar to Bottom-Up Approach, we create a table F to store all $f_{i,j}$ once computed
- However, we modify the recursive algorithm a bit, so that we still solve compute the fastest time in a Top-Down
- Assume: entries of F are initialized empty

Memoization comes from the word "memo"

Original Recursive Algorithm

```
Compute_F(i, j)  /* Finding $f_{i,j}$ */
  1. if (j == 0)  return $p_{i,0}$;
  2. g =  Compute_F(i,j-1) + $p_{i,j}$ ;
  3. h =  Compute_F(3-i,j-1) + $t_{3-i,j-1}$ + $p_{i,j}$ ;
  4. return min { g, h } ;
```

Memoized Version

```
Memo_Compute_F(i, j)  /* Finding $f_{i,j}$ */
    if (j == 0)  return $p_{i,0}$;
    if (F[i,j-1] is empty)
        F[i,j-1] = Memo_Compute_F(i,j-1) ;
    if (F[3-i,j-1] is empty)
        F[3-i,j-1] = Memo_Compute_F(3-i,j-1);
    g =  F[i,j-1] + $p_{i,j}$ ;
    h =  F[3-i,j-1] + $t_{3-i,j-1}$ + $p_{i,j}$ ;
    return min { g, h } ;
```

Memoized Version (Running Time)

To find Memo_Compute_F(1, n):

1. Memo_Compute_F(i, j) is only called when F[i,j] is empty (it becomes nonempty afterwards)
   ➔ Θ(n) calls

2. Each Memo_Compute_F(i, j) call only needs Θ(1) time apart from recursive calls

Running Time = Θ(n)

Dynamic Programming

The previous strategy that applies "tables" is called dynamic programming (DP)

[Here, programming means:
a good way to plan things / to optimize the steps]

- A problem that can be solved efficiently by DP often has the following properties:
  1. Optimal Substructure (allows recursion)
  2. Overlapping Subproblems (allows speed up)

Assembly Line Scheduling

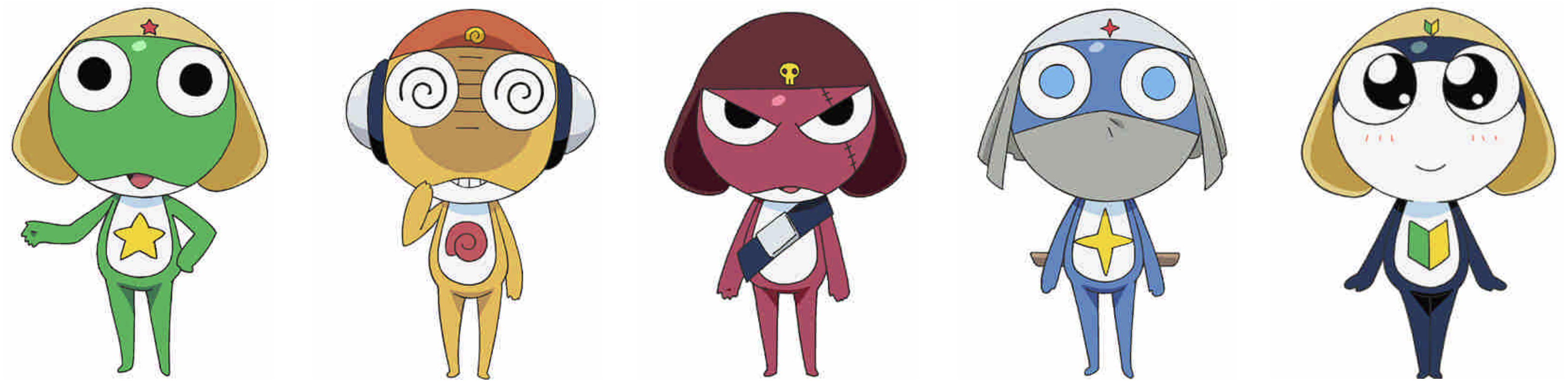
Challenge: We now know how to compute the fastest assembly time. How to get the exact sequence of steps to achieve this time?

Answer: When we compute $f_{i,j}$, we remember whether its value is based on $f_{1,j-1}$ or $f_{2,j-1}$

➔ easy to modify code to get the sequence

Sharing Gold Coins

Five lucky pirates has discovered a treasure chest with 1000 gold coins …



Sharing Gold Coins

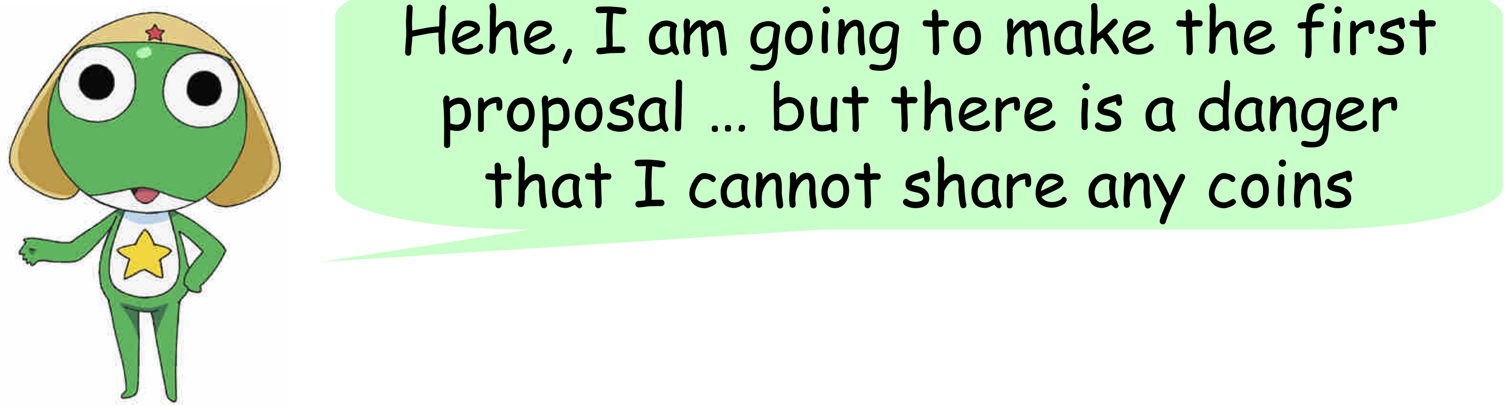
There are rankings among the pirates:

… and they decide to share the gold coins in the following way:

Sharing Gold Coins

First, Rank-1 pirate proposes how to share the coins…

- If at least half of them agree, go with the proposal
- Else, Rank-1 pirate is out of the game



Sharing Gold Coins

If Rank-1 pirate is out, then Rank-2 pirate proposes how to share the coins…

- If at least half of the remaining agree, go with the proposal
- Else, Rank-2 pirate is out of the game

Hehe, I get a chance to propose if Rank-1 pirate is out of the game

Sharing Gold Coins

In general, if Rank-1, Rank-2, …, Rank-k pirates are out, then Rank-(k+1) pirate proposes how to share the coins…

- If at least half of the remaining agree, go with the proposal
- Else, Rank-(k+1) pirate is out of the game

Question: If all the pirates are smart, who will get the most coin? Why?