CS4311
Design and Analysis of Algorithms

Supplement of Lecture 5:
Probability & Expectation

About this lecture

- What is Probability?
- What is an Event?
- What is a Random Variable?
- What is Expectation or "Average Value" of a Random Variable?
- Useful Thm: Linearity of Expectation

Experiment and Sample Space

- An experiment is a process that produces an outcome
- A random experiment is an experiment whose outcome is not known until it is observed
  - Exp 1: Throw a die once
  - Exp 2: Flip a coin until Head comes up

Experiment and Sample Space

- A sample space Ω of a random experiment is the set of all outcomes
  - Exp 1: Throw a die once
  - Sample space: { 1, 2, 3, 4, 5, 6 }
  - Exp 2: Flip a coin until Head comes up
  - Sample space: ??
- Any subset of sample space Ω is called an event

Probability

- Probability studies the chance of each event occurring
- Informally, it is defined with a function Pr that satisfies the following:

(1) For any event E, $0 \leq Pr(E) \leq 1$

(2) $Pr(\Omega) = 1$

(3) If E_1 and E_2 do not have common outcomes,

$$Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$$

Example

Questions:

1. Suppose the die is a fair die, so that
   $Pr(1) = Pr(2) = \ldots = Pr(6)$.
   What is $Pr(1)$? Why?

2. Instead, if we know
   $Pr(1) = 0.2$, $Pr(2) = 0.3$, $Pr(3) = 0.4$,
   $Pr(4) = 0.1$, $Pr(5) = Pr(6) = 0$.
   What is $Pr(\{1,2,4\})$?

Random Variable

Definition: *A* random variable X on a sample space Ω is a function that maps each outcome of Ω into a real number. That is, X: $\Omega \rightarrow \mathcal{R}$.

Ex: Suppose that we throw two dice

➔ Ω = { (1,1), (1,2), …, (6,5), (6,6) }

Define X = sum of outcome of two dice

➔ X is a random variable on Ω

Random Variable

- For a random variable X and a value a, the notation

  "X = a"

  denotes the set of outcomes ω in the sample space such that X(ω) = a

  ➔ "X = a" is an event

- In previous example,

  "X = 10" is the event {(4,6), (5,5), (6,4)}

Expectation

Definition: The expectation (or average value) of a random variable X, is

$$E[X] = \sum_i i \Pr(X=i)$$

Question:

- X = sum of outcomes of two fair dice
  What is the value of E[X] ?
- How about the sum of three dice?

Expectation (Example)

Let X = sum of outcomes of two dice.

The value of X can vary from 2 to 12

So, we calculate:

$Pr(X=2) = 1/36$, $Pr(X=3) = 2/36$,

$Pr(X=4) = 3/36$, … , $Pr(X=12) = 2/36$,

$$E[X] = 2*Pr(X=2) + 3*Pr(X=3) + \ldots + 11*Pr(X=11) + 12*Pr(X=12) = 7$$

Linearity of Expectation

Theorem: Given random variables $X_1, X_2, \ldots, X_k$, each with finite expectation, we have

$$E[X_1+X_2+\ldots+X_k] = E[X_1]+E[X_2] + \ldots + E[X_k]$$

Let X = sum of outcomes of two dice.

Let X_i = the outcome of the i^{th} dice

What is the relationship of X, X_1, and X_2?

Can we compute $E[X]$?

Linearity of Expectation (Example)

Let X = sum of outcomes of two dice.

Let X_i = the outcome of the i^{th} dice

➔ $X = X_1 + X_2$

➔ $E[X] = E[X_1+X_2] = E[X_1] + E[X_2]$

$= 7/2 + 7/2 = 7$

Can you compute the expectation of the sum of outcomes of three dice?