# CS4311
# Design and Analysis of Algorithms

## Lecture 4: Heapsort

# About this lecture

- Introduce Heap
  - Shape Property and Heap Property
  - Heap Operations
- Heapsort: Use Heap to Sort
- Fixing heap property for all nodes
- Use Array to represent Heap
- Introduce Priority Queue

# Heap

A heap (or binary heap) is a binary tree that satisfies both:

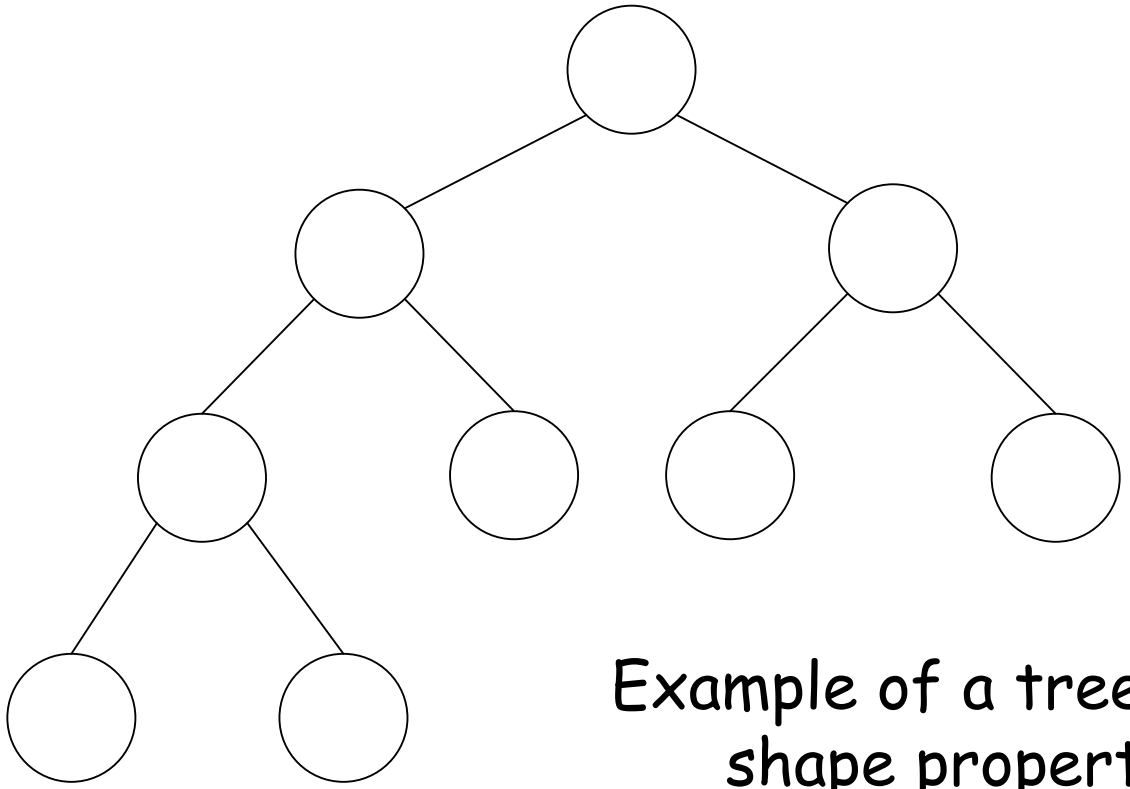
(1) Shape Property

- All levels, except deepest, are fully filled
- Deepest level is filled from left to right

(2) Heap Property

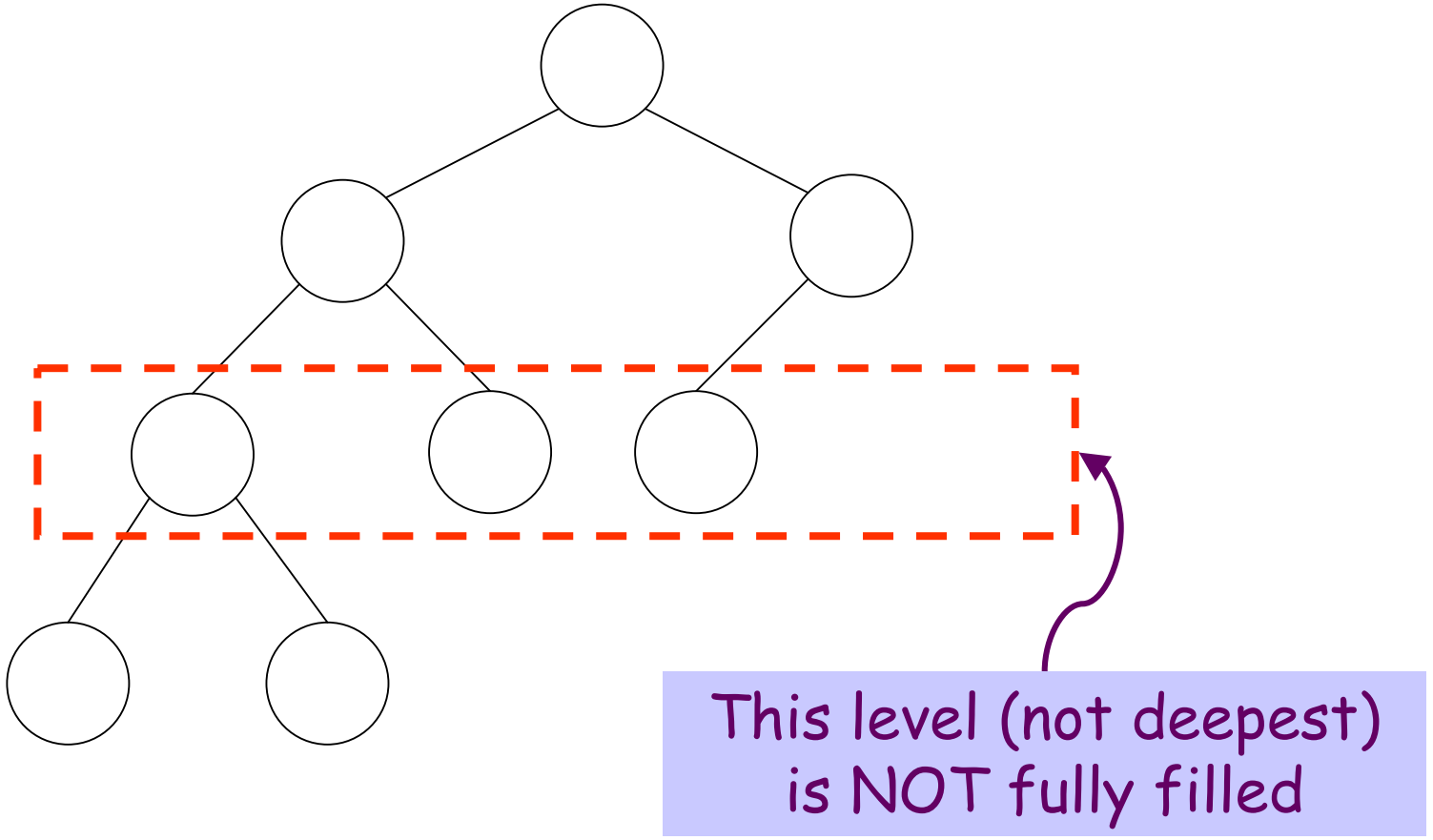
- Value of a node $\leq$ Value of its children

# Satisfying Shape Property

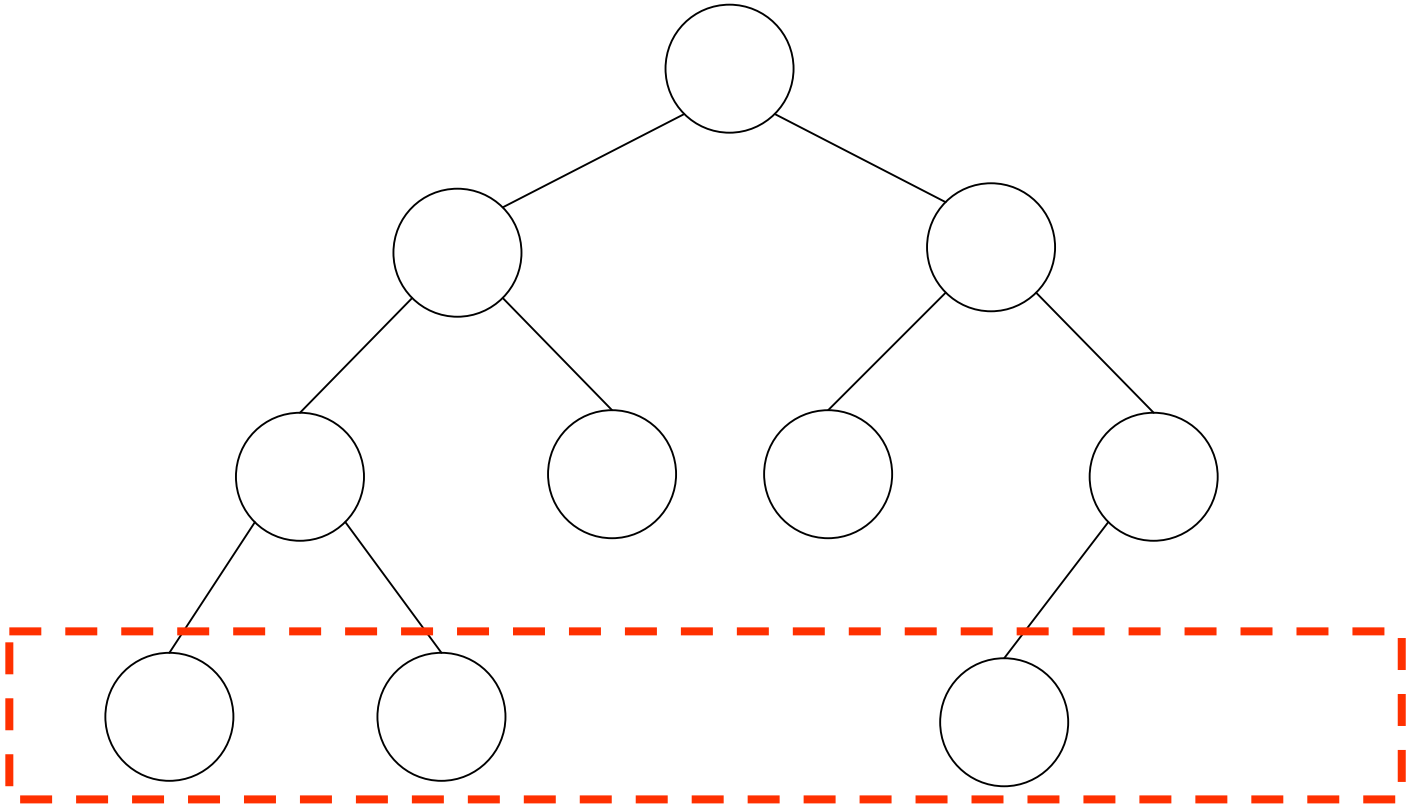


Example of a tree with shape property

# Not Satisfying Shape Property

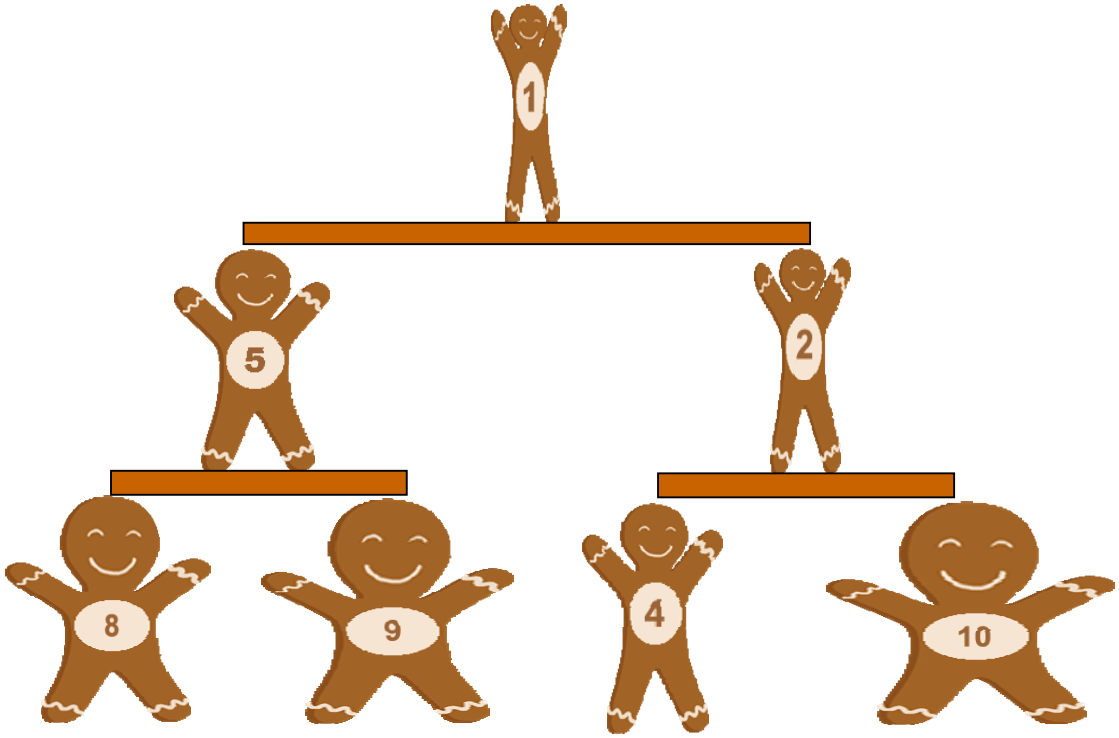


# Not Satisfying Shape Property

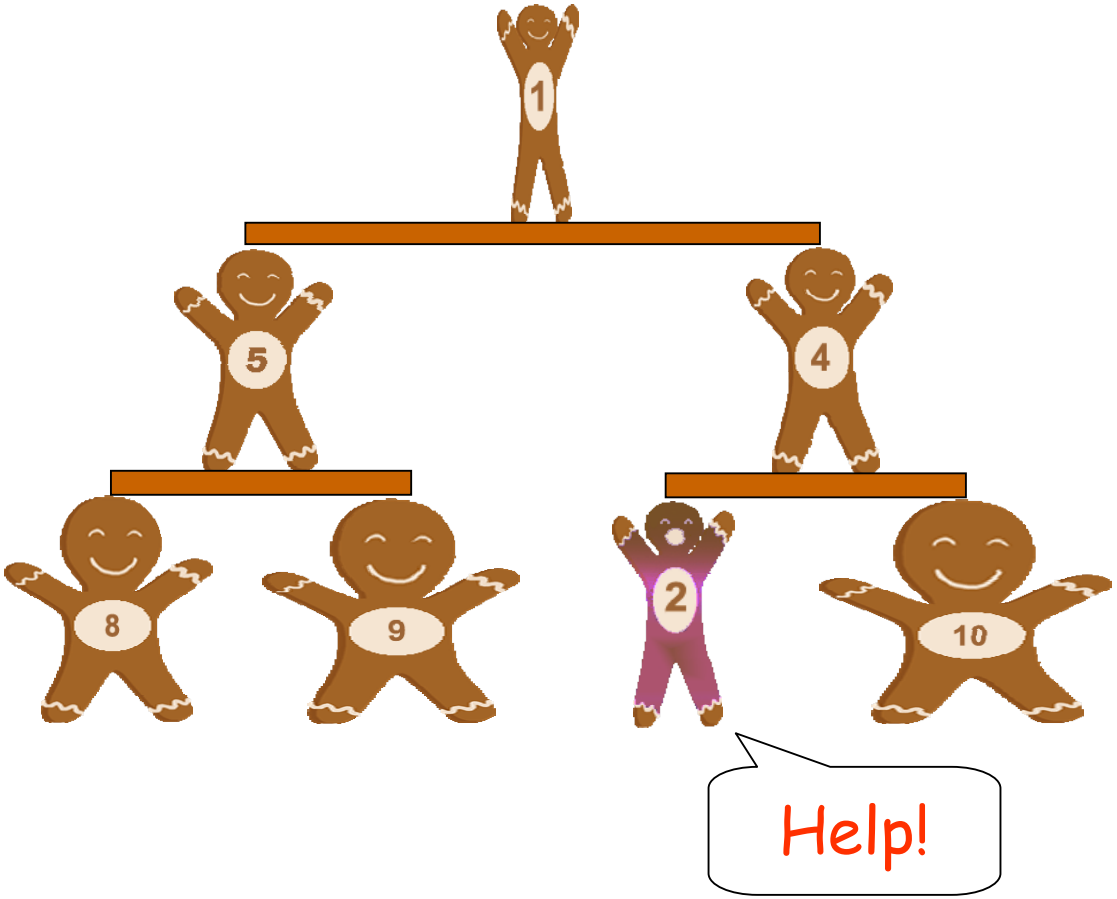


Deepest level NOT
filled from left to right

# Satisfying Heap Property



# Not Satisfying Heap Property



# Min Heap

Q. Given a heap, what is so special about the root's value?

A. … always the minimum

Because of this, the previous heap is also called a min heap

# Heap Operations

- Find-Min : find the minimum value

  ➔ $\Theta(1)$ time

- Extract-Min : delete the minimum value

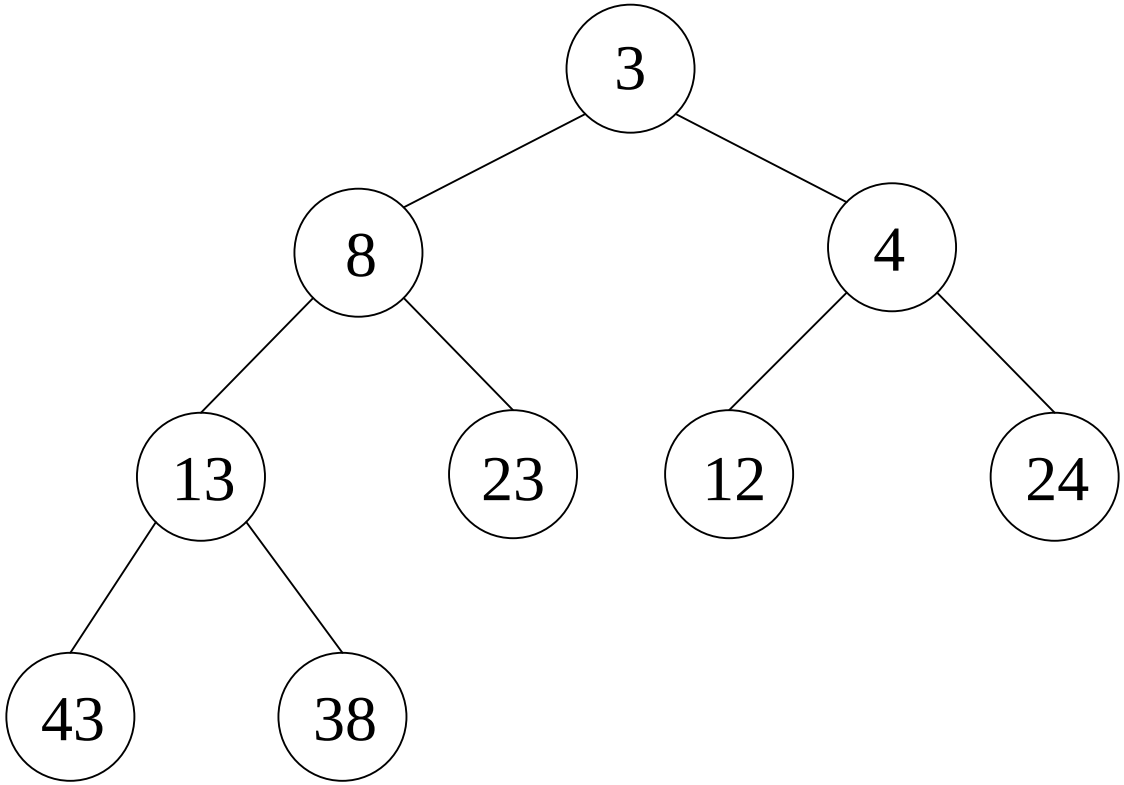
  ➔ $O(\log n)$ time (how??)

- Insert : insert a new value into heap

  ➔ $O(\log n)$ time (how??)

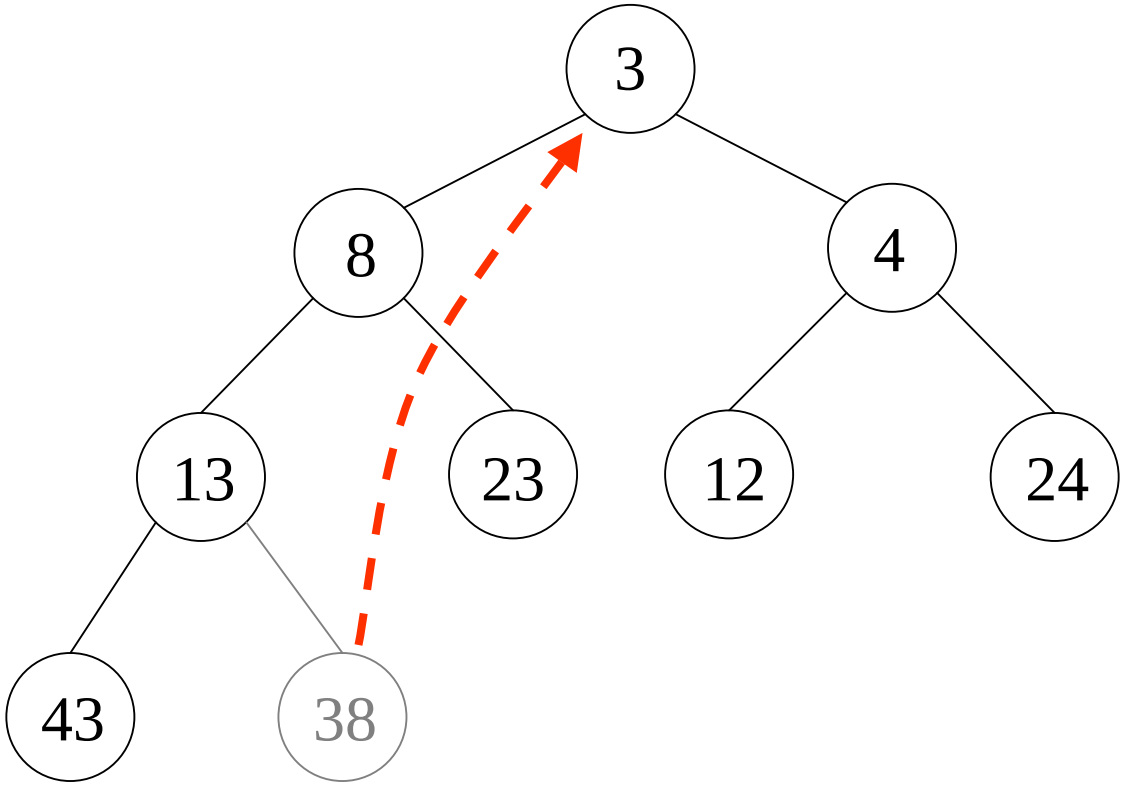
n = # nodes in the heap

# How to do Extract-Min?



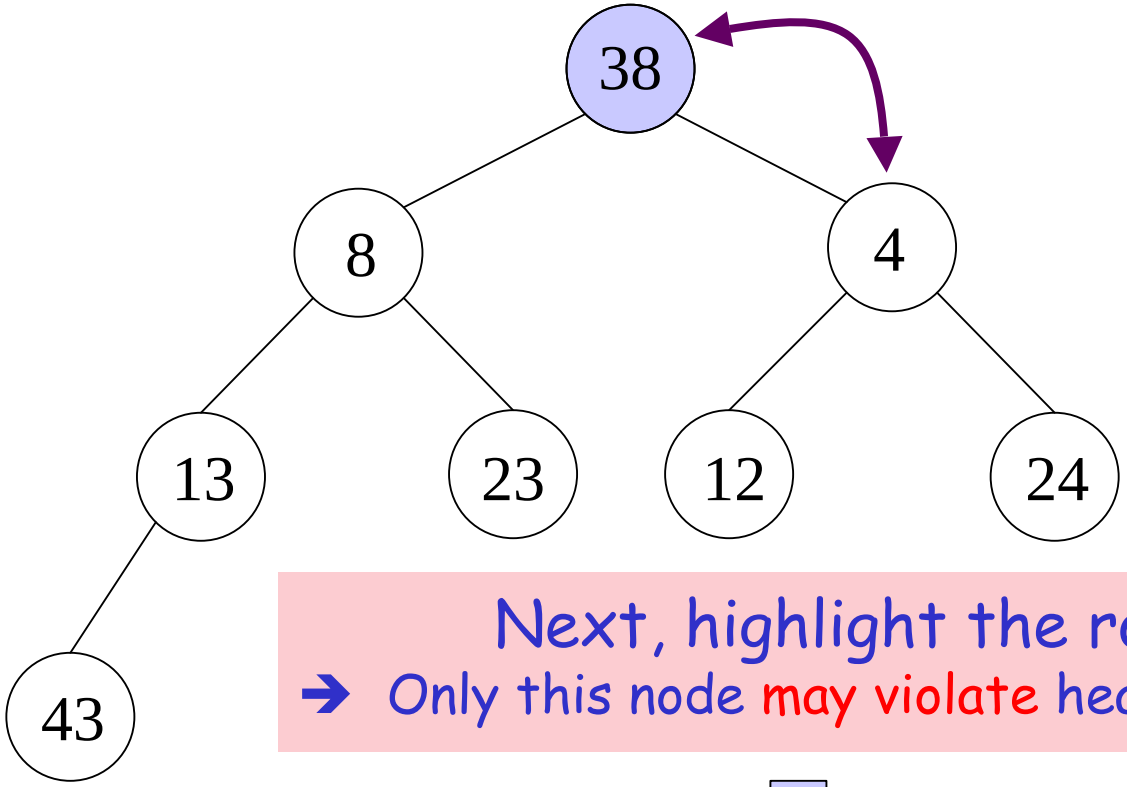

Heap before Extract-Min

# Step 1: Restore Shape Property




Copy value of last node to root.
Next, remove last node

# Step 2: Restore Heap Property

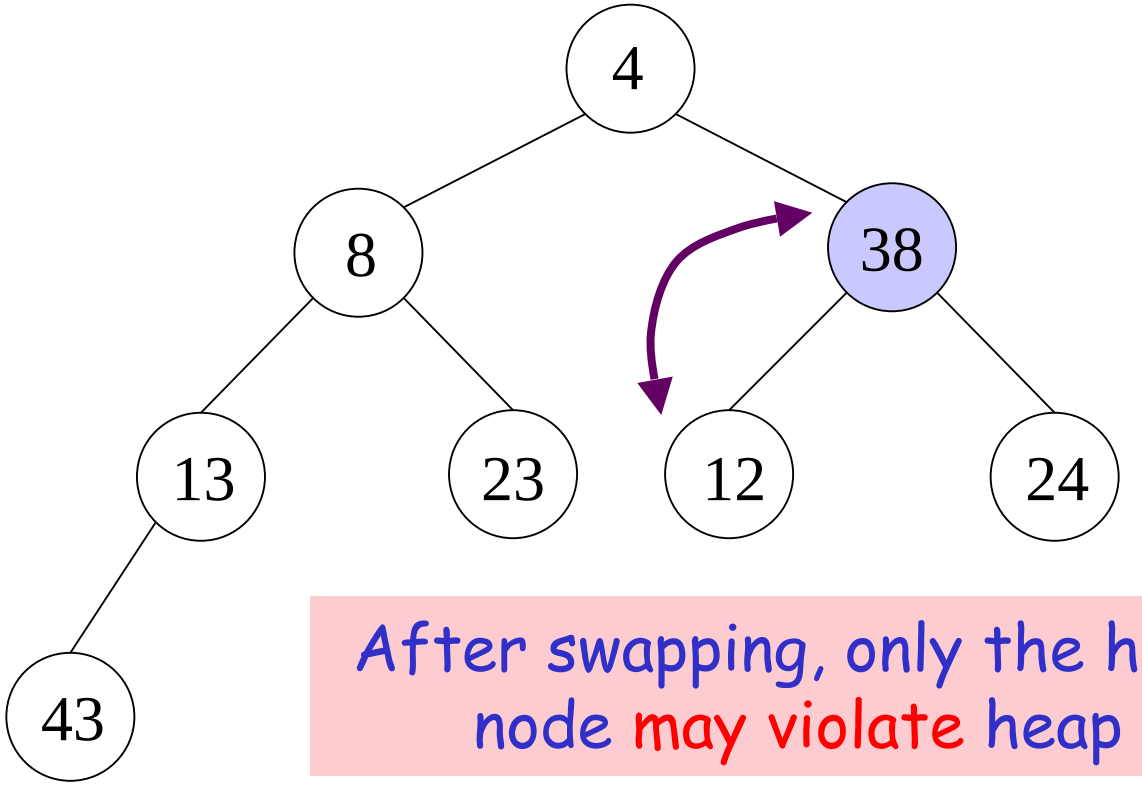



Next, highlight the root

➔ Only this node may violate heap property

If violates, swap highlighted node with "smaller" child
(if not, everything done)

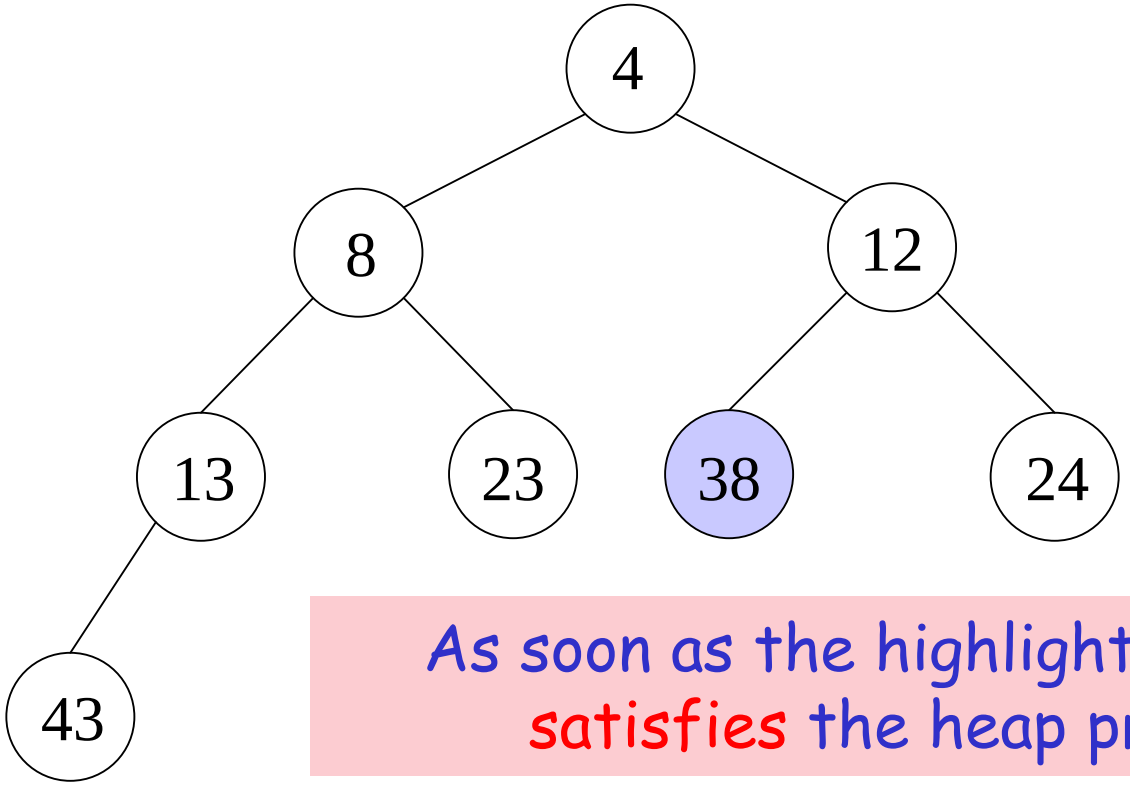
# Step 2: Restore Heap Property




After swapping, only the highlighted node may violate heap property

If violates, swap highlighted node with "smaller" child (if not, everything done)

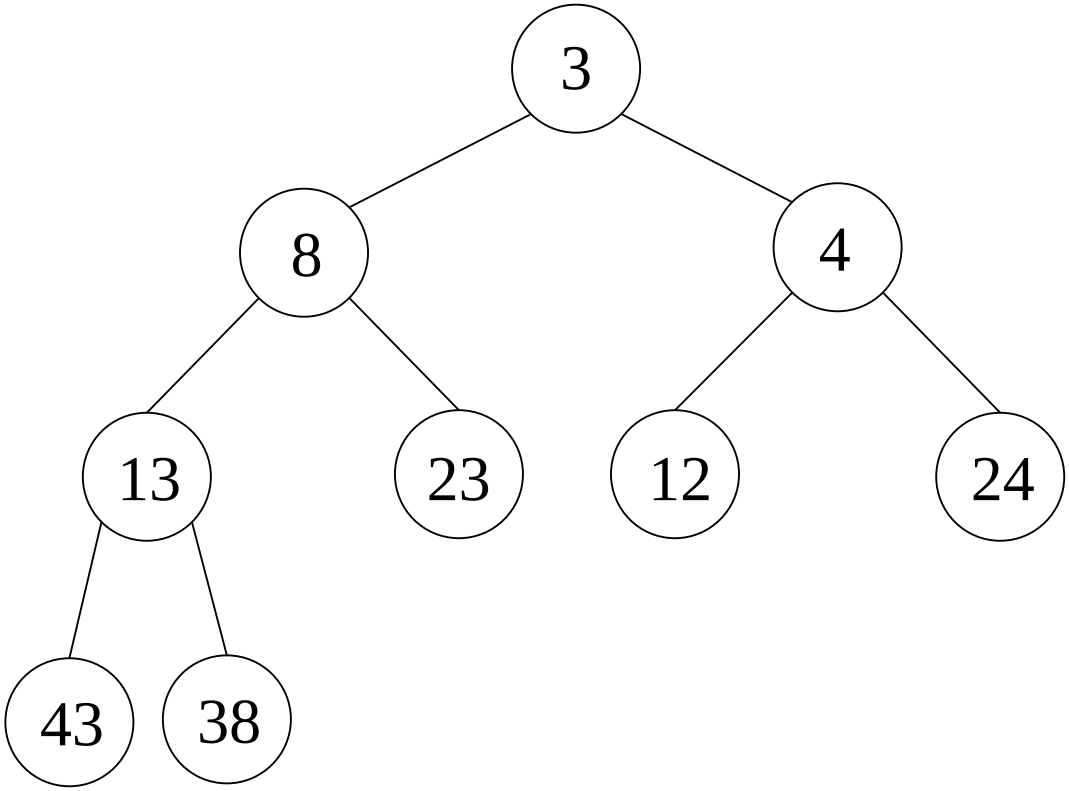
# Step 2: Restore Heap Property




As soon as the highlighted node satisfies the heap property

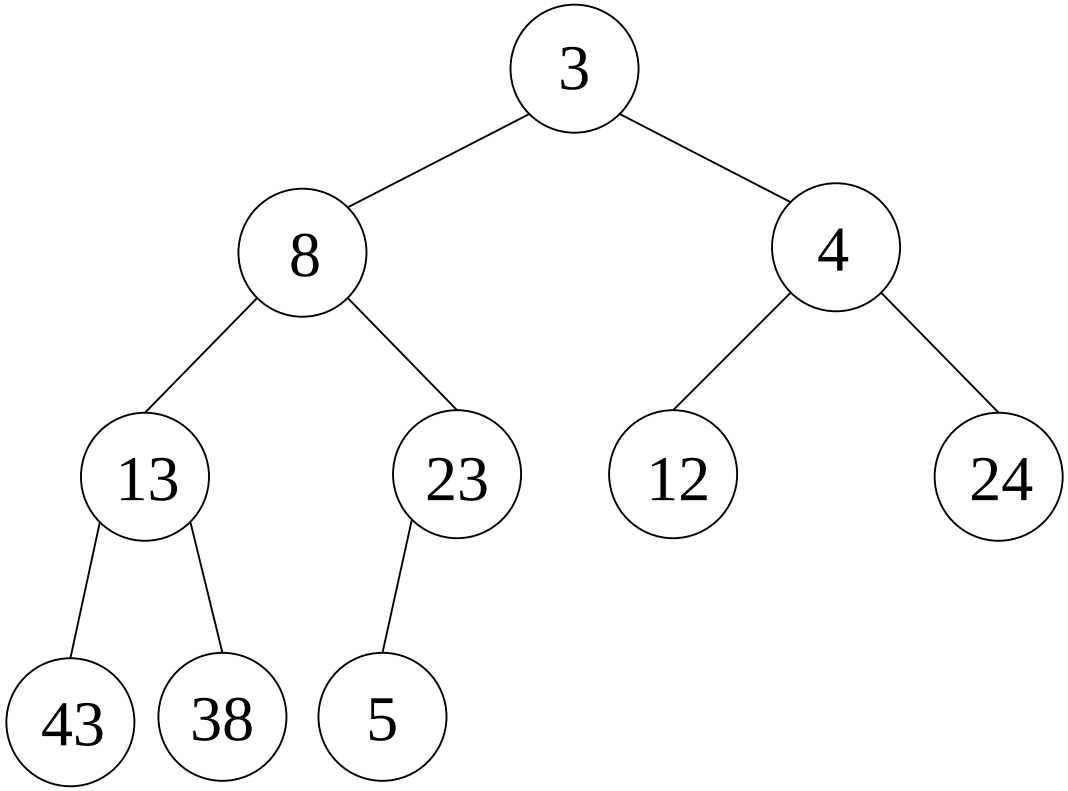
Everything done !!!

# How to do Insert?



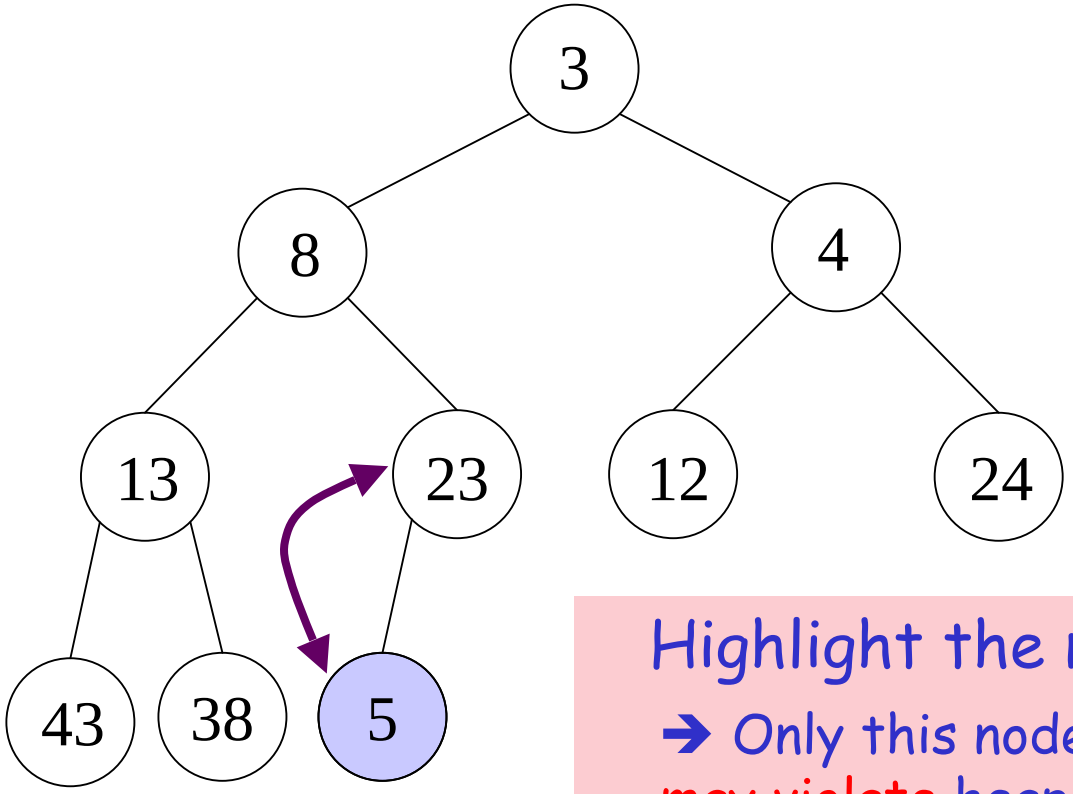

Heap before Insert

# Step 1: Restore Shape Property




Create a new node with the new value.
Next, add it to the heap at correct position

# Step 2: Restore Heap Property

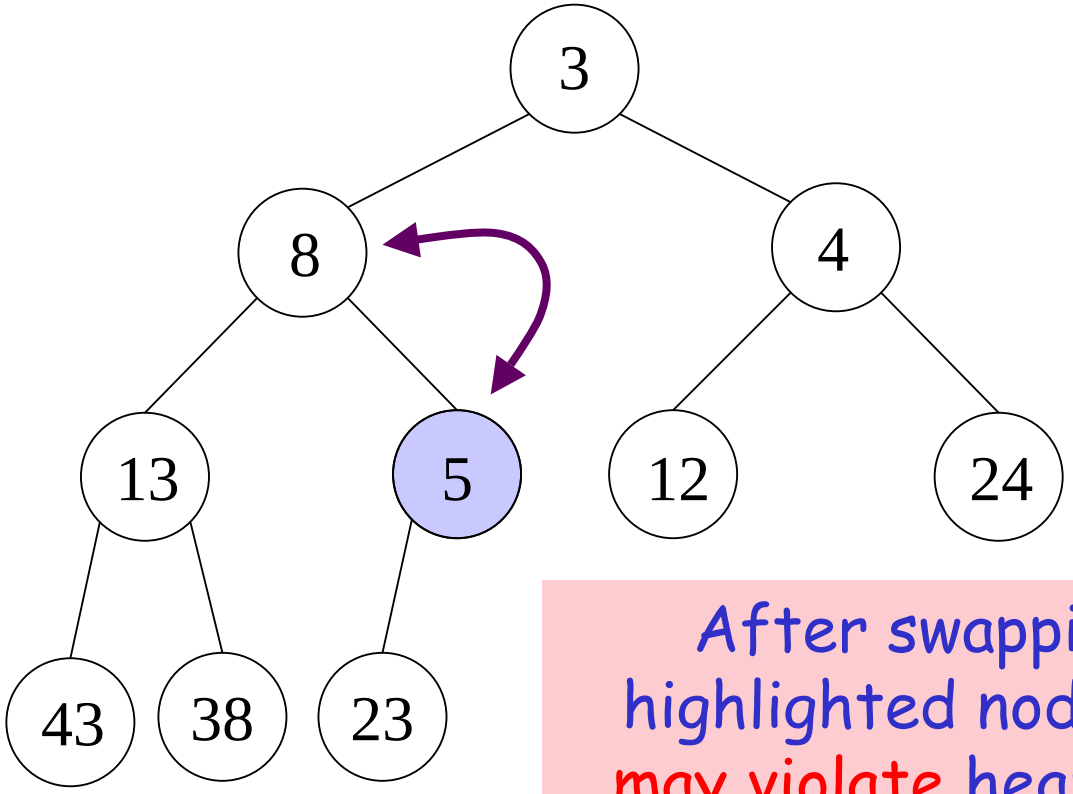



Highlight the new node

➔ Only this node's parent may violate heap property

If violates, swap highlighted node with parent
(if not, everything done)

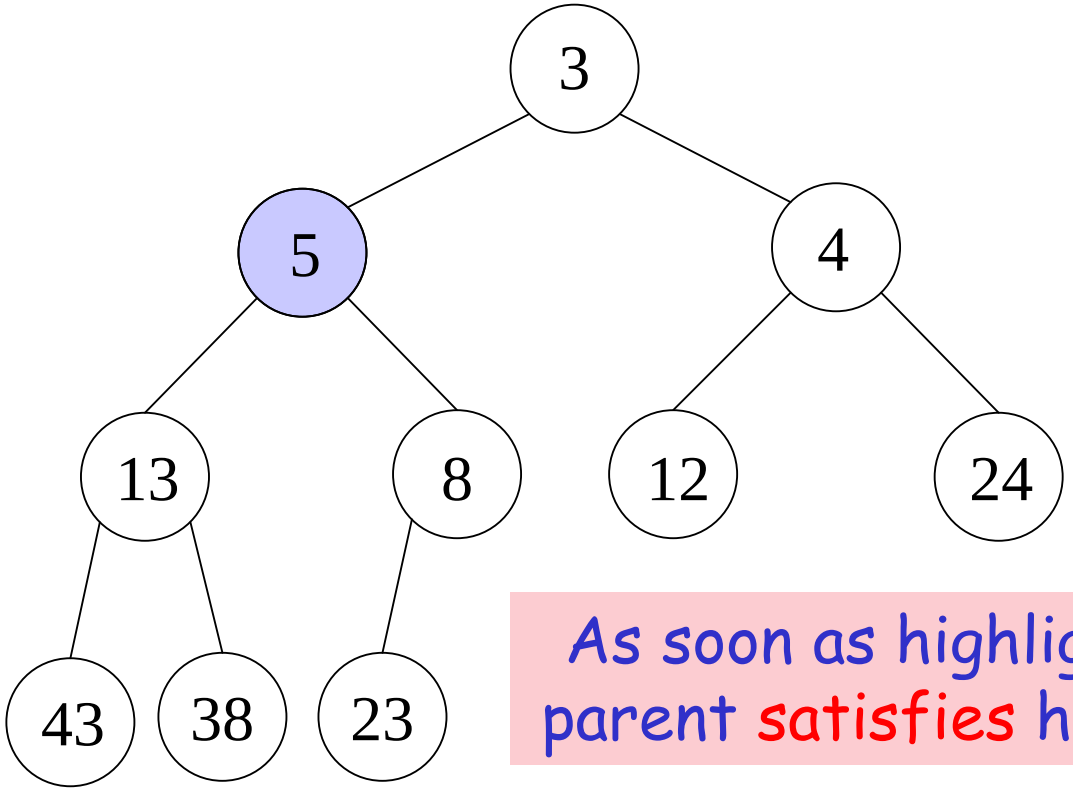
# Step 2: Restore Heap Property



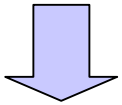

After swapping, only highlighted node's parent may violate heap property

If violates, swap highlighted node with parent
(if not, everything done)

# Step 2: Restore Heap Property




As soon as highlighted node's parent satisfies heap property



Everything done !!!

# Running Time

Let $h$ = height of heap

- Both Extract-Min and Insert require $O(h)$ time to perform

Since $h = \Theta(\log n)$ (why??)

➔ Both require $O(\log n)$ time

$n$ = # nodes in the heap

# Heapsort

Q. Given n numbers, can we use heap to sort them, say, in ascending order?

A. Yes, and extremely easy !!!

1. Call Insert to insert n numbers into heap
2. Call Extract-Min n times

   ➔ numbers are output in sorted order

Runtime: $n \times O(\log n) + n \times O(\log n) = O(n \log n)$

This sorting algorithm is called heapsort

# Challenge
## (Fixing heap property for all nodes)

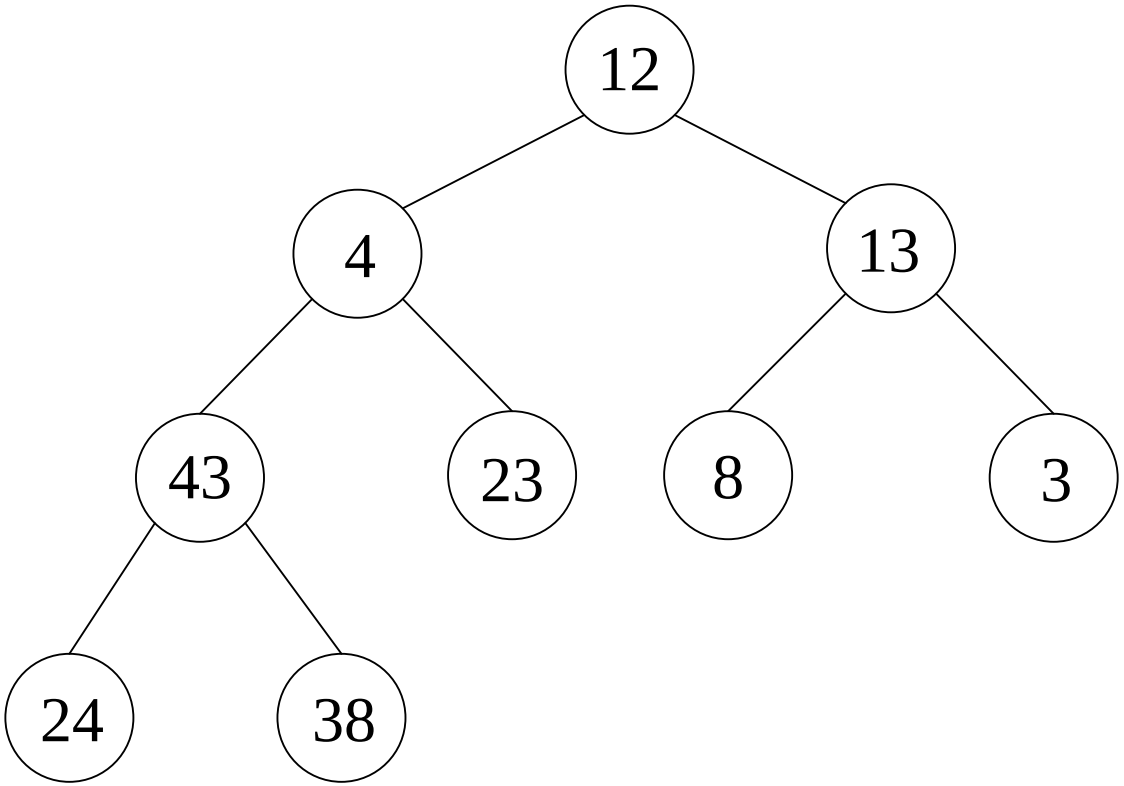
Suppose that we are given a binary tree which satisfies the shape property

However, the heap property of the nodes may not be satisfied …

Question: Can we make the tree into a heap in $O(n)$ time?

n = # nodes in the tree

# How to make it a heap?

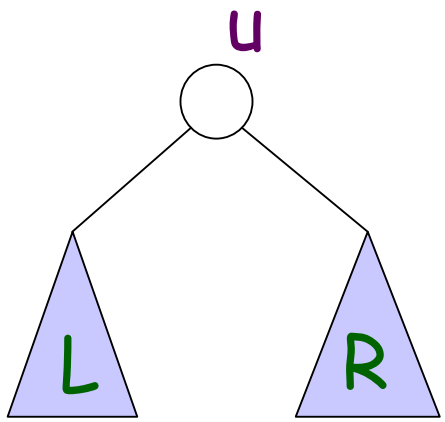


# Observation

u = root of a binary tree

L = subtree rooted at u's left child

R = subtree rooted at u's right child




Obs: If L and R satisfy heap property, we can make the tree rooted at u satisfy heap property in $O(\max\{\text{height}(L), \text{height}(R)\})$ time.

We denote the above operation by Heapify(u)

# Heapify

Then, for any tree T, we can make T satisfy the heap property as follows:

Step 1. h = height(T) ;

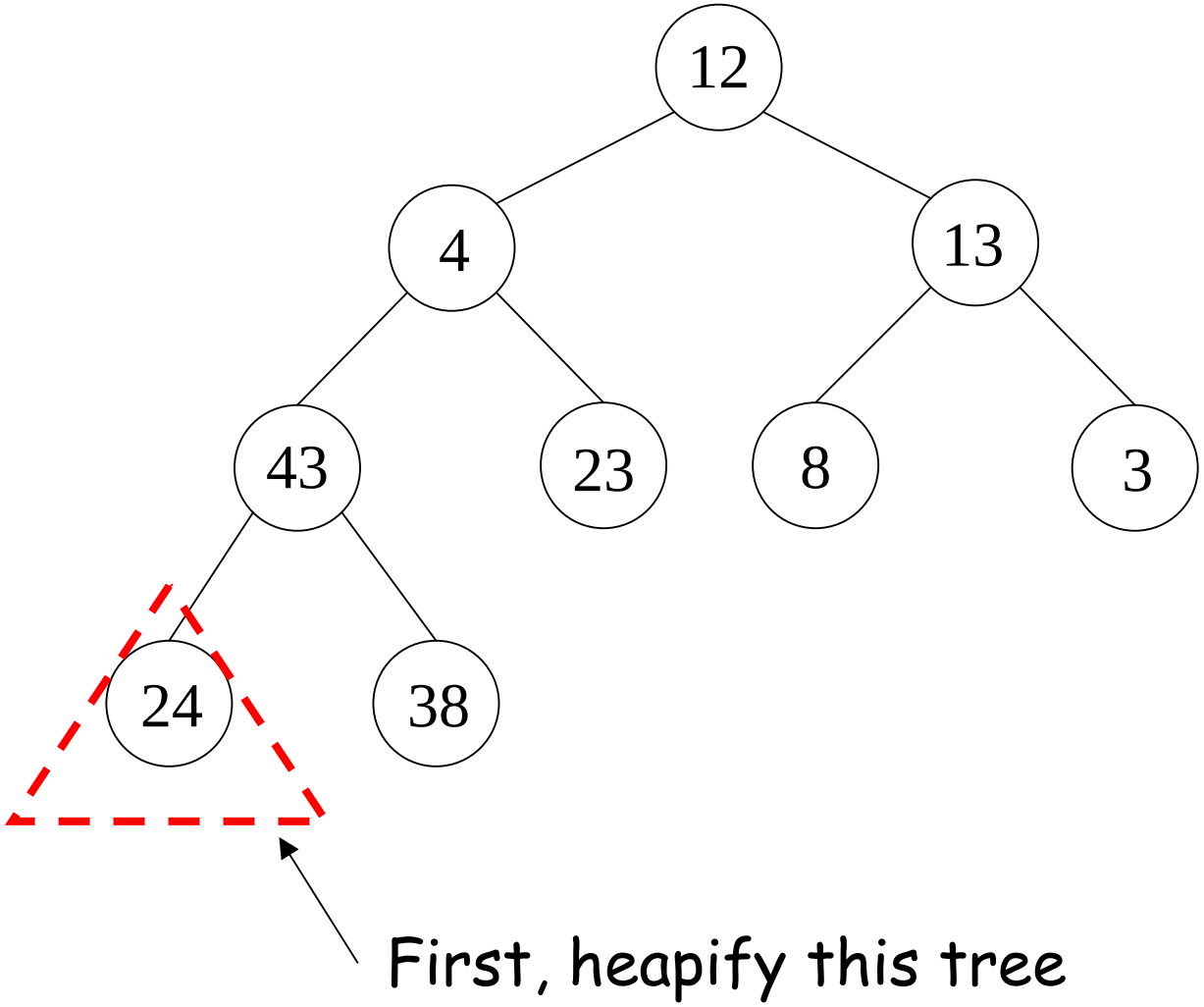
Step 2. for k = h, h-1, …, 1

for each node u at level k

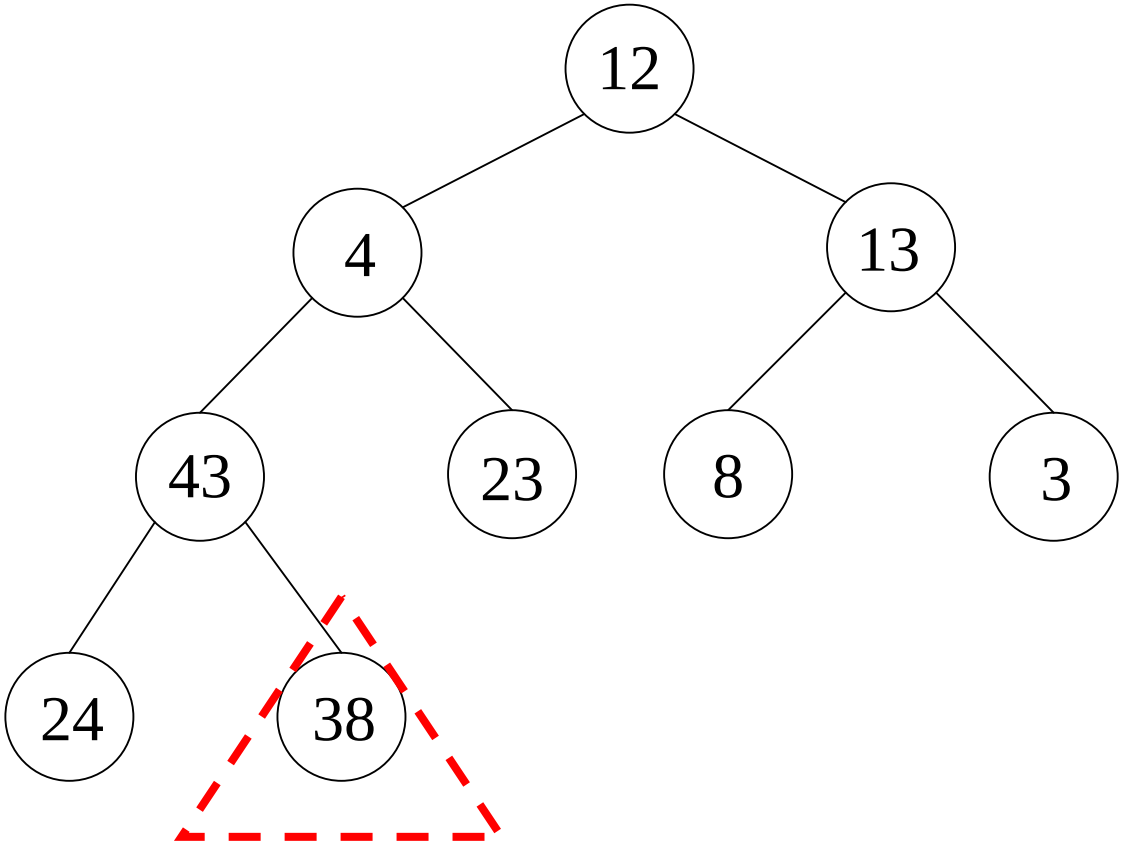
Heapify(u) ;

Why is the above algorithm correct?

# Example Run



# Example Run



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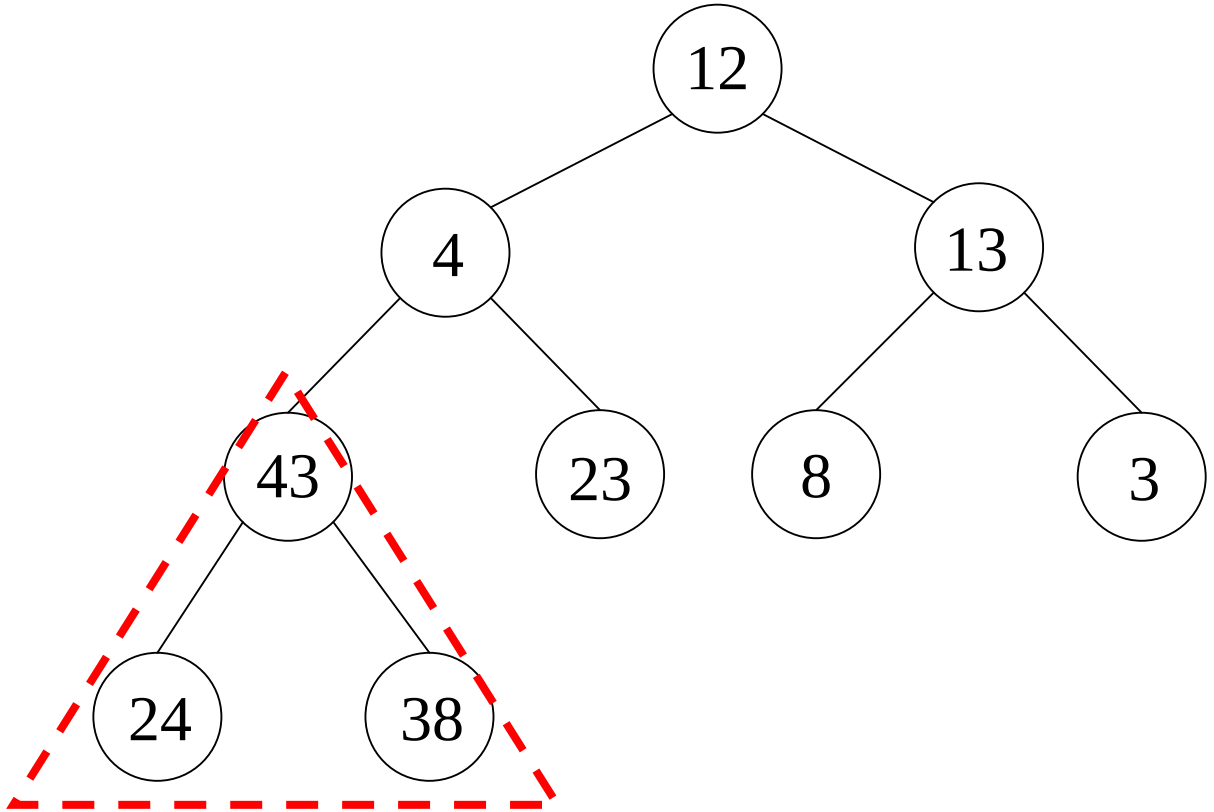
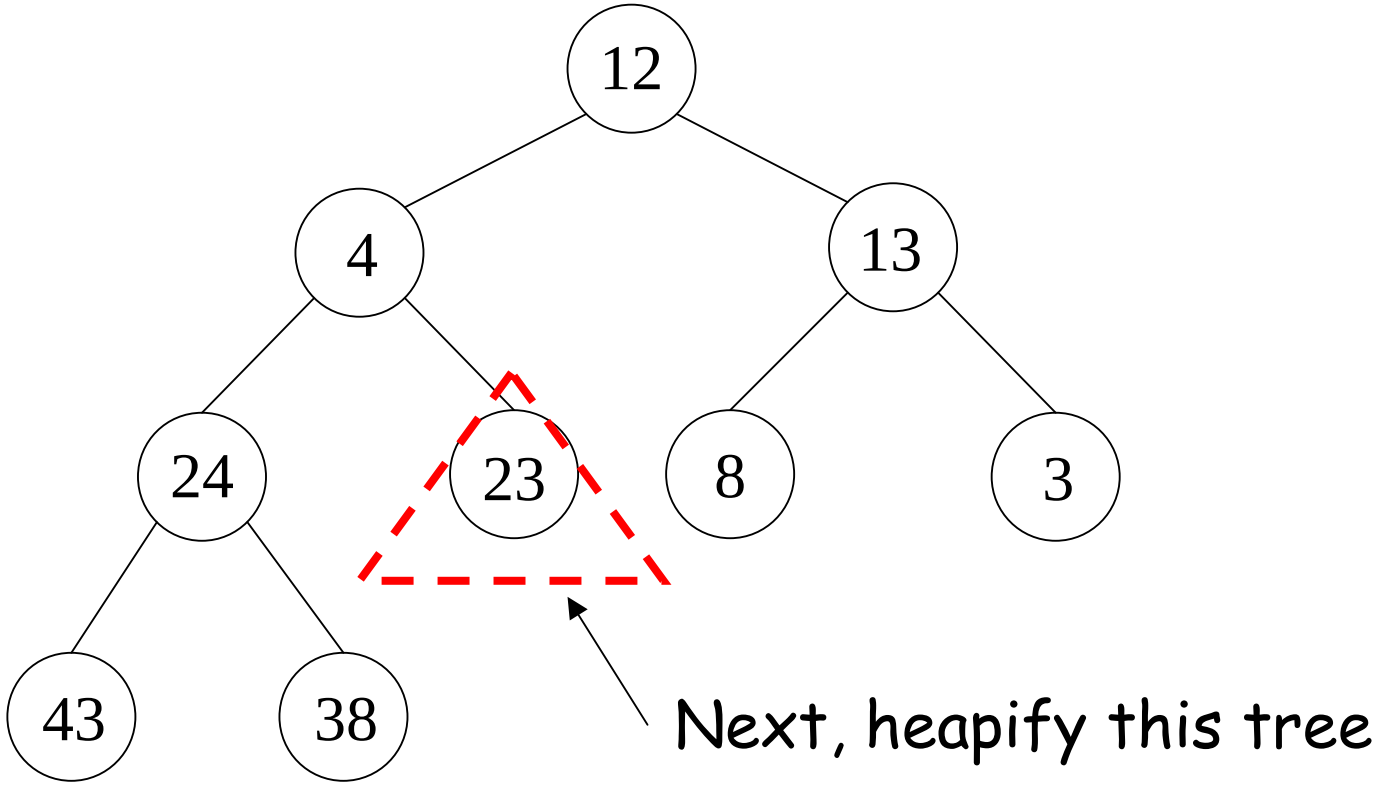


# Example Run



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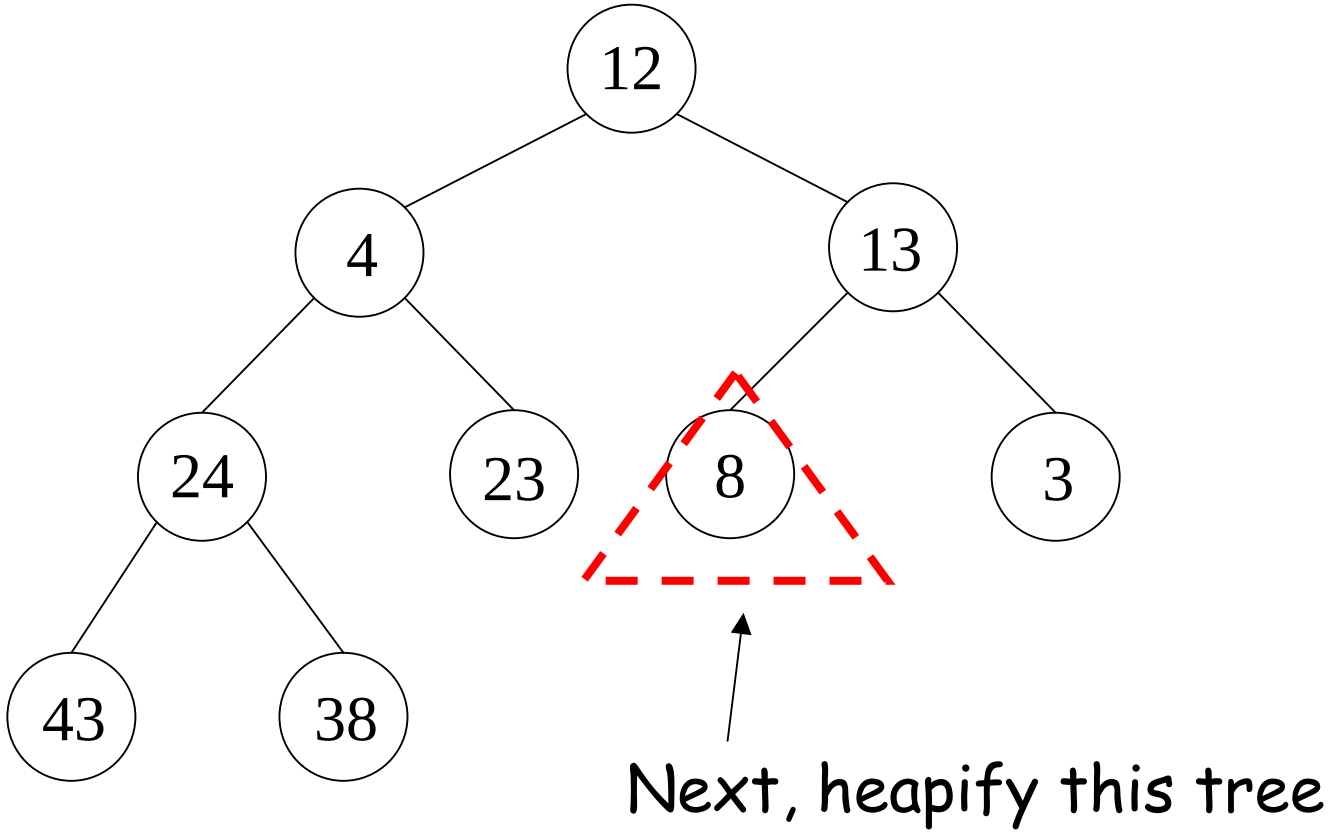

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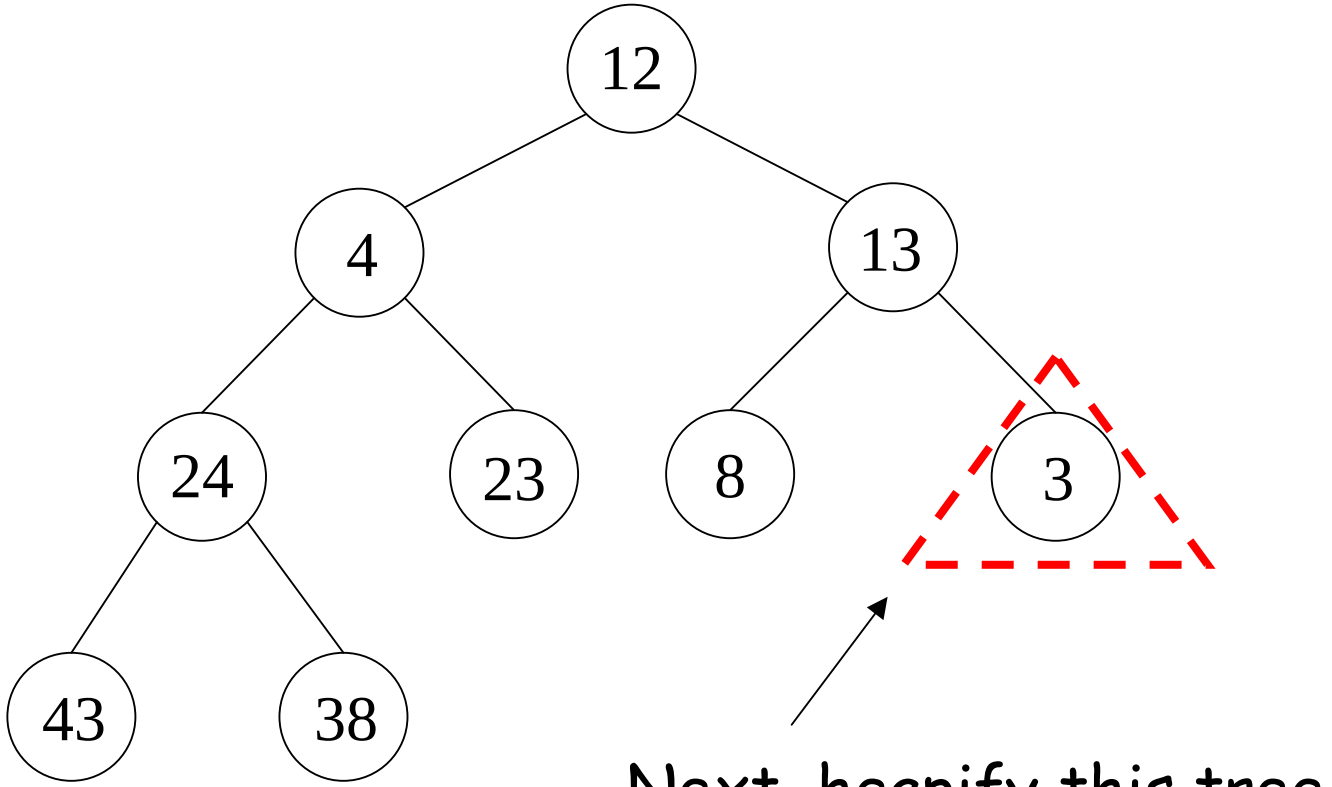

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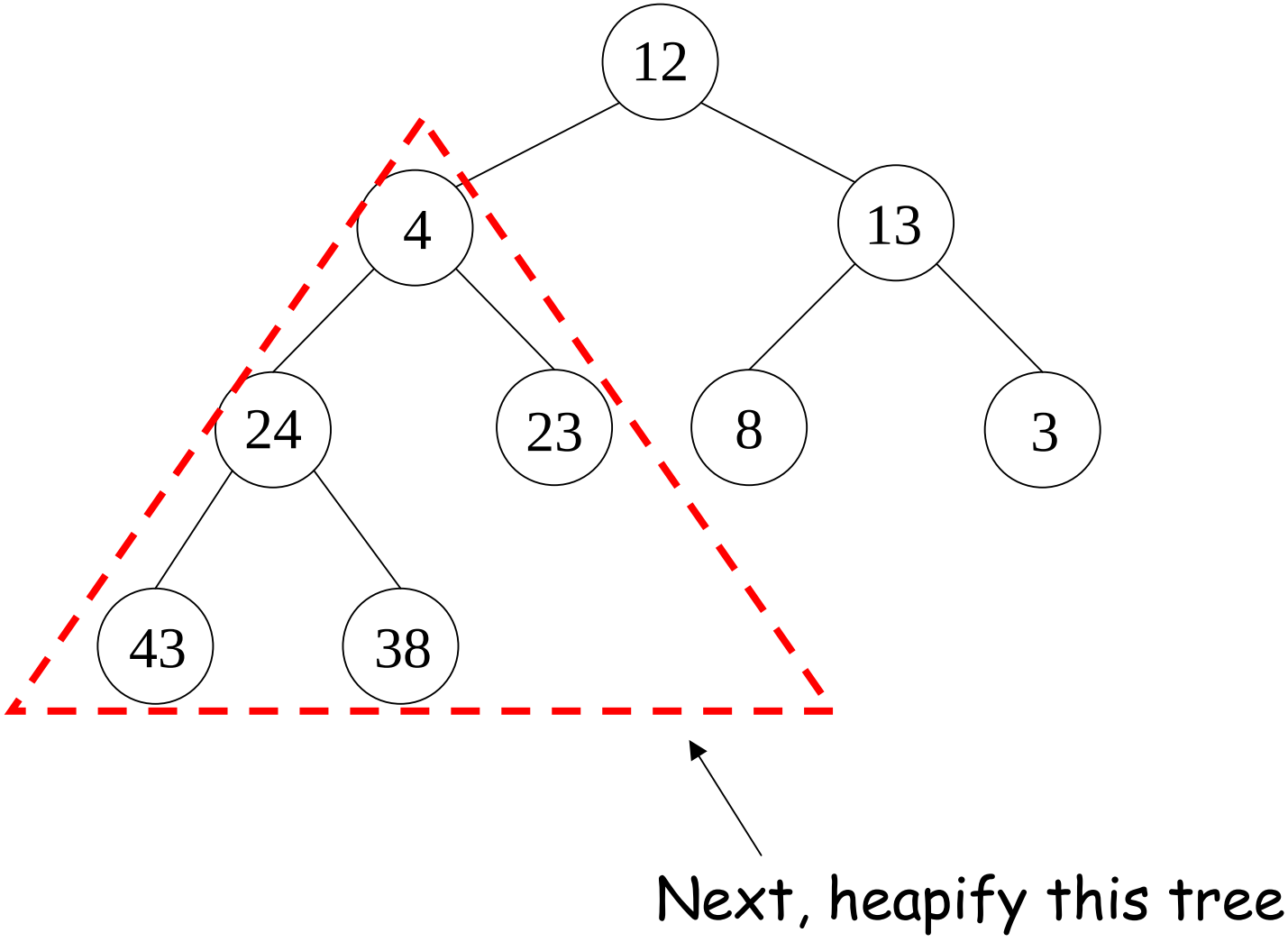

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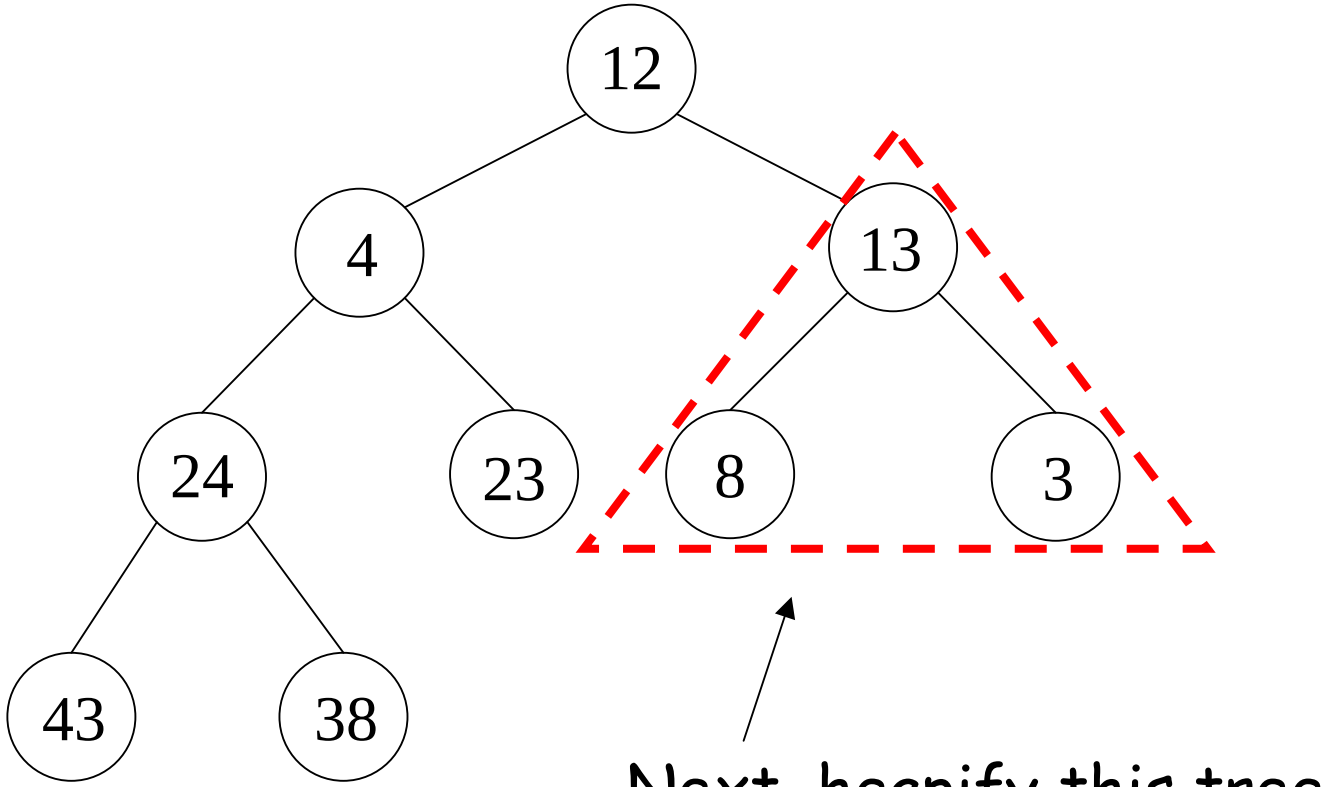
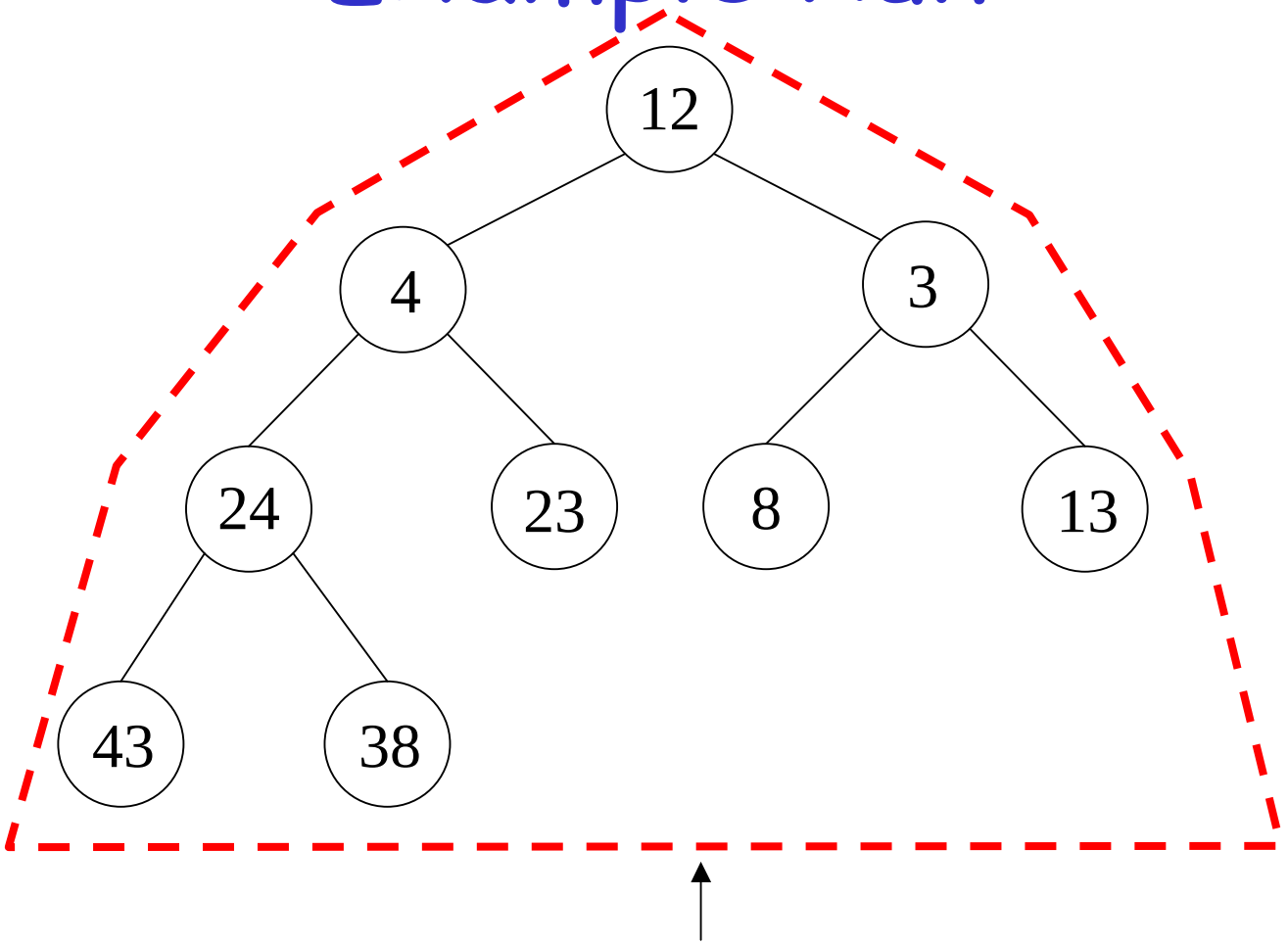
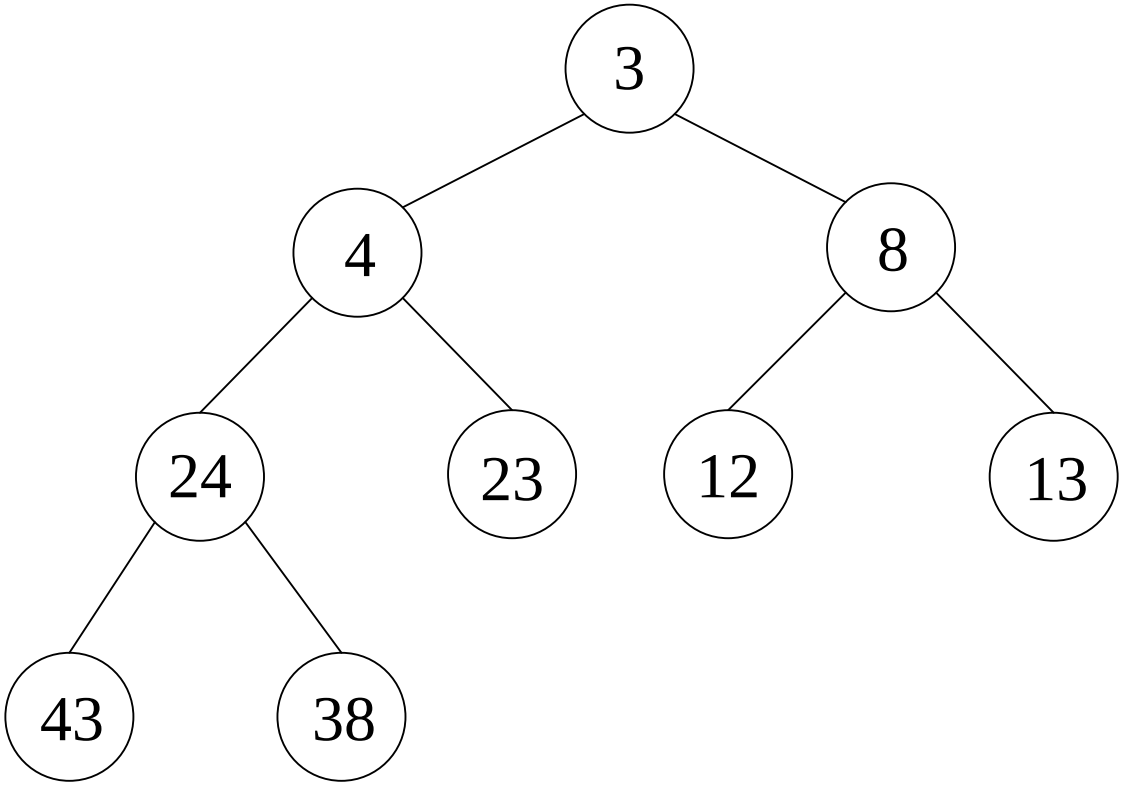


# Example Run




Finally, heapify the whole tree

# Example Run




Everything Done !

# Back to the Challenge
## (Fixing heap property for all nodes)

Suppose that we are given a binary tree which satisfies the shape property

However, the heap property of the nodes may not be satisfied …

Question: Can we make the tree into a heap in $O(n)$ time?

n = # nodes in the tree

# Back to the Challenge
## (Fixing heap property for all nodes)

Let **h** = height of tree

So, $2^{h-1} \leq n \leq 2^h - 1$ (why??)

For a tree with shape property,

at most $2^{h-1}$ nodes at level h,

exactly $2^{h-2}$ nodes at level h-1,

exactly $2^{h-3}$ nodes at level h-2, …

# Back to the Challenge
## (Fixing heap property for all nodes)

Using the previous algorithm to solve the challenge, the total time is at most

$$2^{h-1}\times 1 + 2^{h-2}\times 2 + 2^{h-3}\times 3 + \ldots + 1\times h \quad \text{[why??]}$$

$$= 2^h \left( 1\times\tfrac{1}{2} + 2\times(\tfrac{1}{2})^2 + 3\times(\tfrac{1}{2})^3 + \ldots + h\times(\tfrac{1}{2})^h \right)$$

$$\leq 2^h \sum_{k=1 \text{ to } \infty} k\times(\tfrac{1}{2})^k = 2^h \times 2 \leq 4n$$

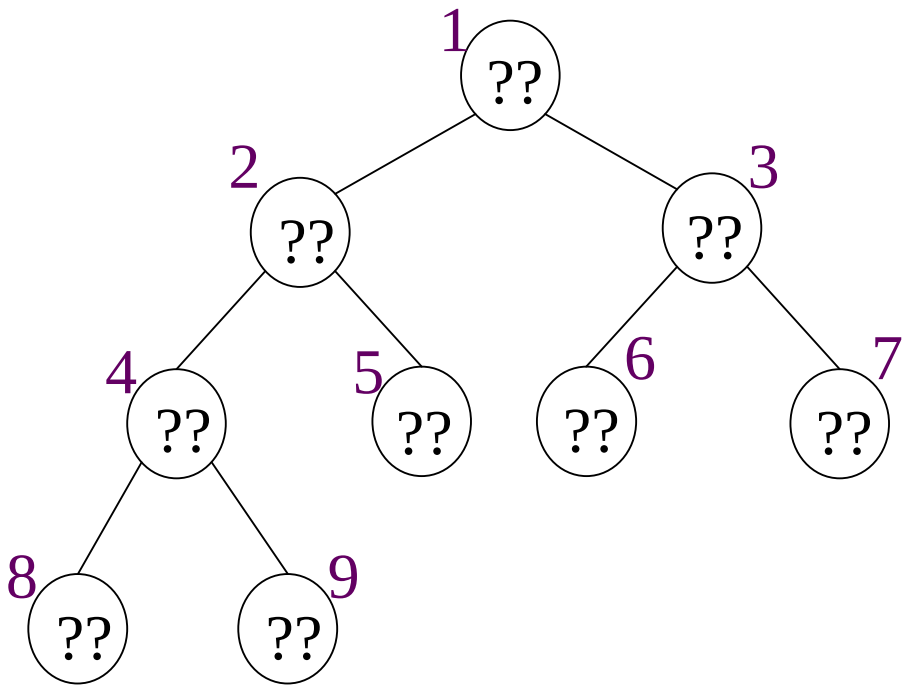
➔ Thus, total time is $O(n)$

# Array Representation of Heap

Given a heap.

Suppose we mark the position of root as 1, and mark other nodes in a way as shown in the right figure. (BFS order)

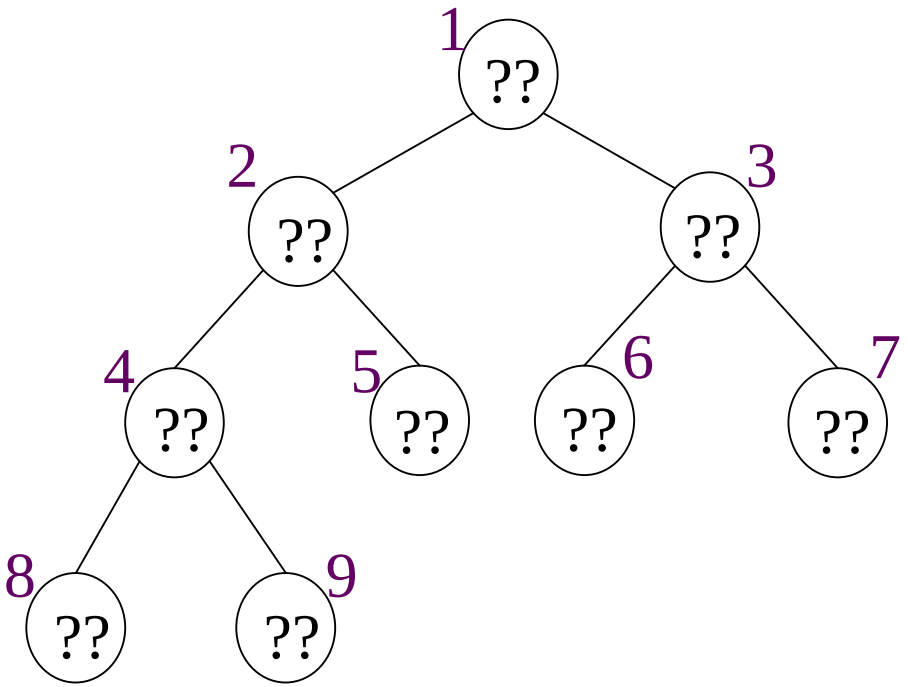
Anything special about this marking?



# Array Representation of Heap

Yes, something special:

1. If the heap has n nodes, the marks are from 1 to n
2. Children of x, if exist, are 2x and 2x+1
3. Parent of x is $\lfloor x/2 \rfloor$



# Array Representation of Heap

The special properties about the marking immediately gives the following:

We can use an array $A[1..n]$ to represent a heap of size $n$

Advantage:
Avoid storing or using tree pointers !!

# Max Heap

We can also define a max heap, by changing the heap property to:

Value of a node $\geq$ Value of its children

Max heap supports the following operations:

(1) Find Max, (2) Extract Max, (3) Insert

Do you know how to do these operations?

# Priority Queue

Let S = a set of items, each has a numeric key value

Priority queue on S is a data structure that supports the following operations:

Min(S): return item with min key

Extract-Min(S): return and remove item with min key

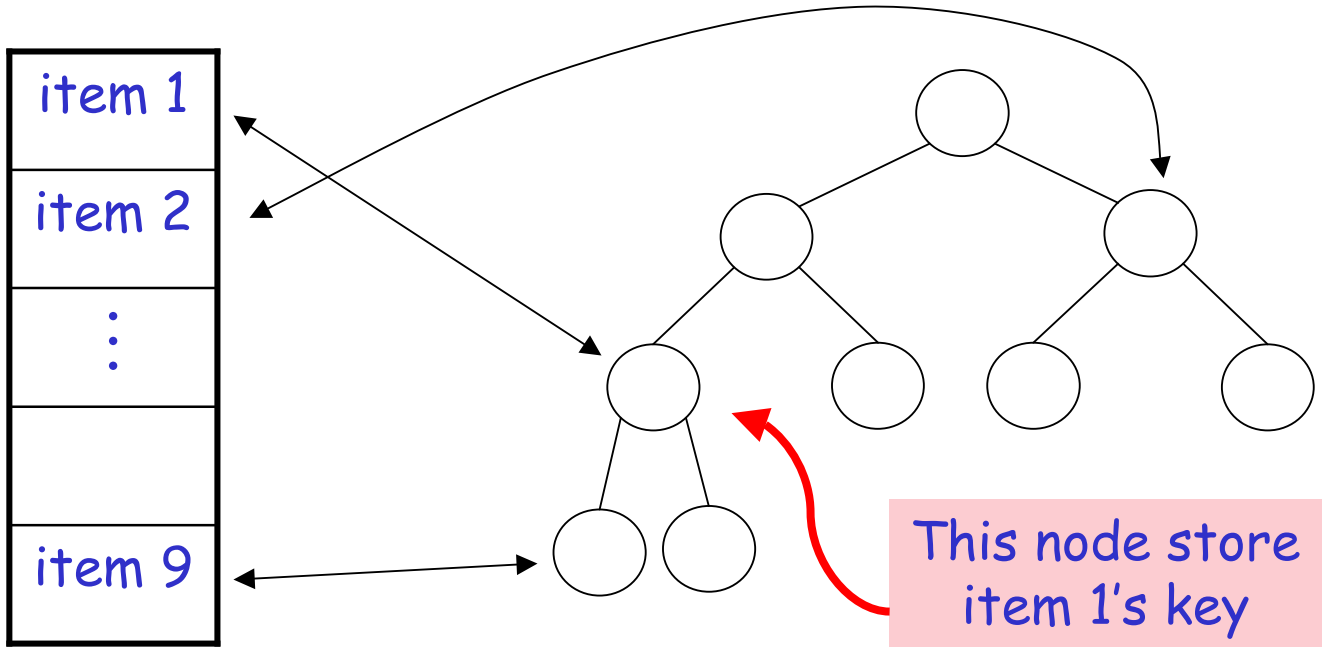
Insert(S,x,k): insert item x with key k into S

Decrease-Key(S,x,k): decrease key of item x to k

# A Scheme for Priority Queue

1. Store the items in a table.
2. Use a heap to store keys of the items.
3. Store links between an item and its key

E.g.,



# A Scheme for Priority Queue

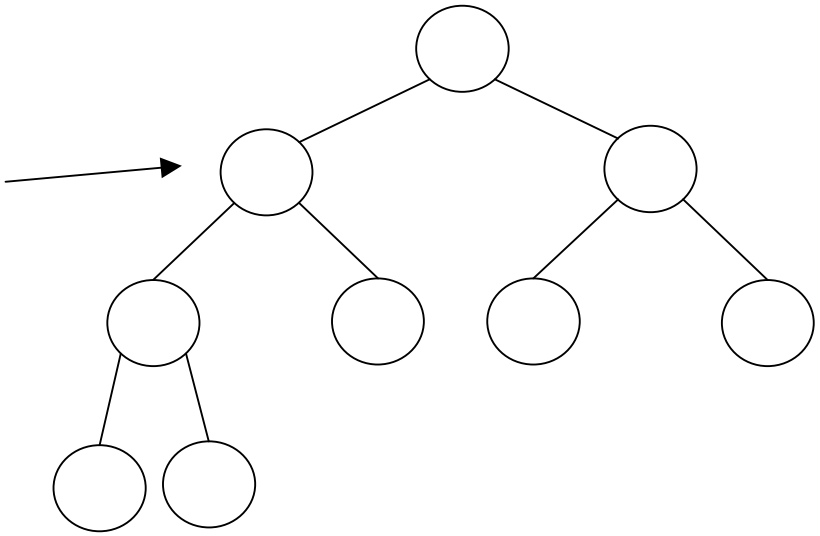
The previous scheme can immediately support Min, Extract-Min, and Insert.

Also it can support Decrease-Key in O(log n) time.

E.g.,

Node storing key value of item x

How do we decrease the key to k ??



# Other Schemes?

In later lectures, we will look at other ways to implement a priority queue

- Basically, they will have some trade-off between efficiency of the operations

Remark: Min-Priority Queue is useful in MST or shortest path algorithm

Remark: Max-Priority Queue supports Max, Extract-Max, Insert, and Increase-Key and is useful in job scheduling