# CS4311
# Design and Analysis of Algorithms

## Lecture 27:

## Single-Source Shortest-Path

# About this lecture

- What is the problem about ?
- Dijkstra's Algorithm [1959]
  - ~ Prim's Algorithm [1957]
- Folklore Algorithm for DAG [???]
- Bellman-Ford Algorithm
  - Discovered by Bellman [1958], Ford [1962]
  - Allowing negative edge weights

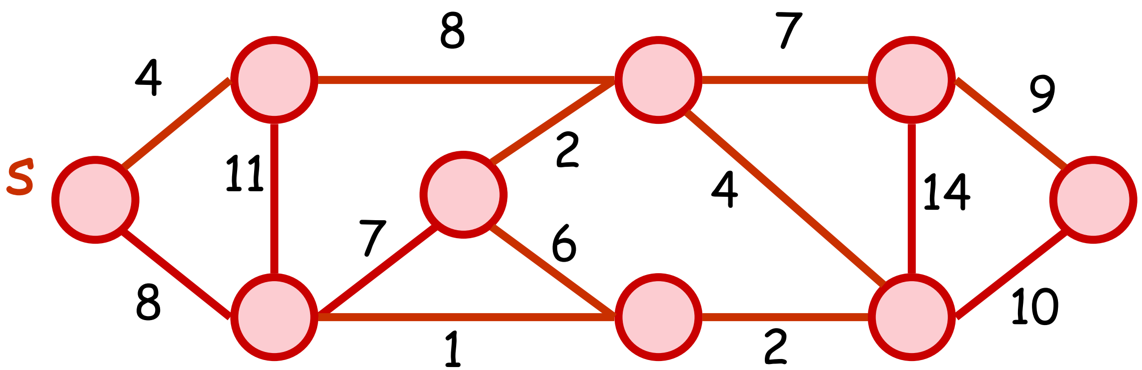
# Single-Source Shortest Path

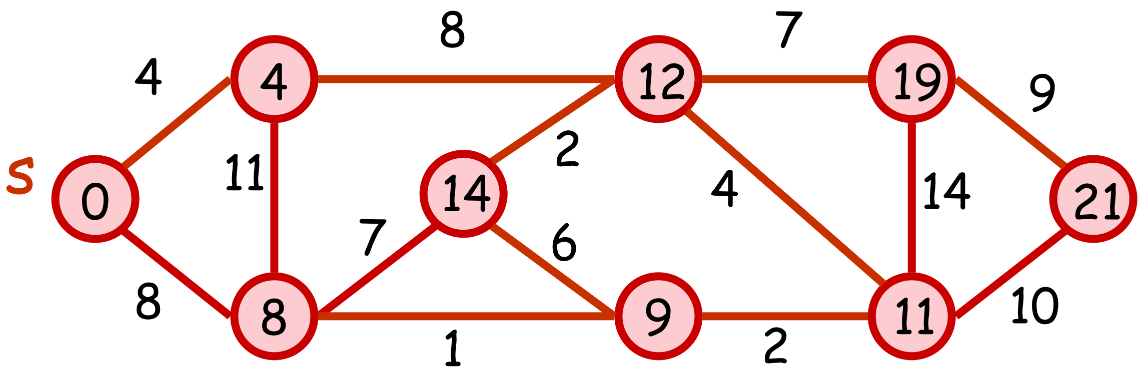
- Let $G = (V,E)$ be a weighted graph
  - the edges in $G$ have weights
  - can be directed/undirected
  - can be connected/disconnected
- Let $s$ be a special vertex, called source

Target: For each vertex $v$, compute the length of shortest path from $s$ to $v$

# Single-Source Shortest Path

- E.g.,

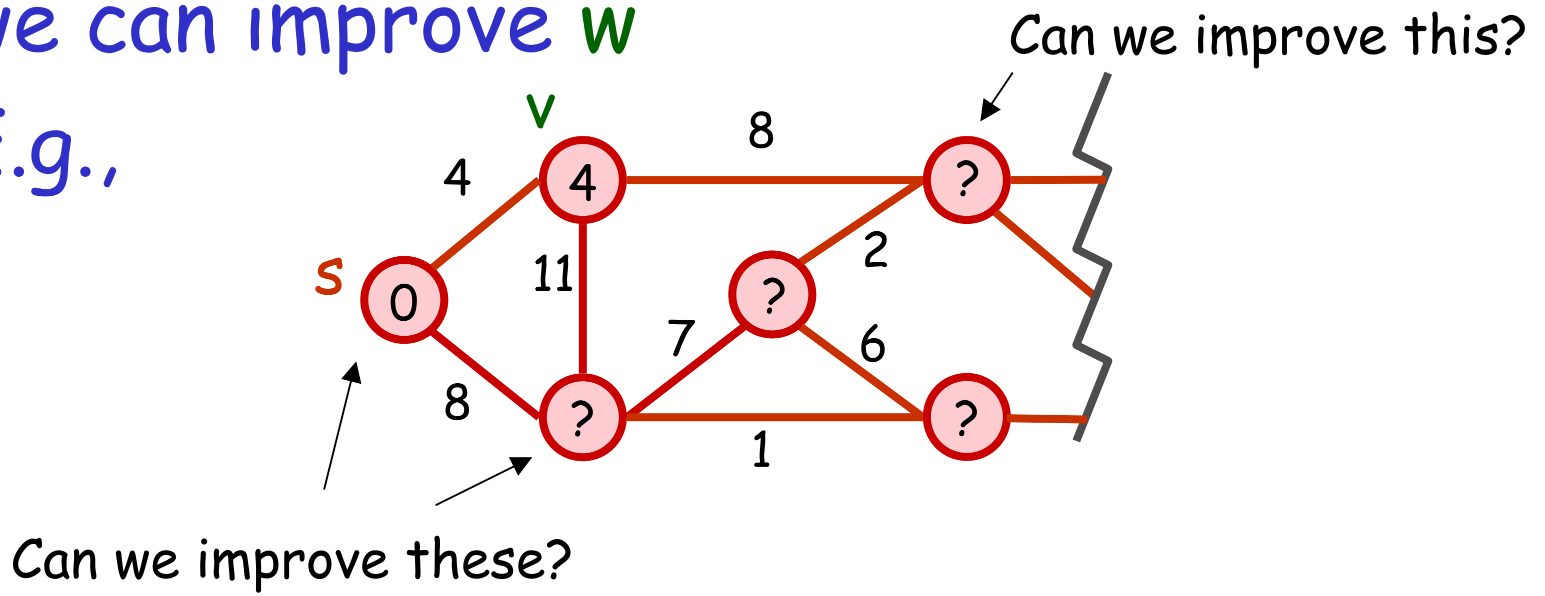


# Relax

- A common operation that is used in the three algorithms is called Relax :

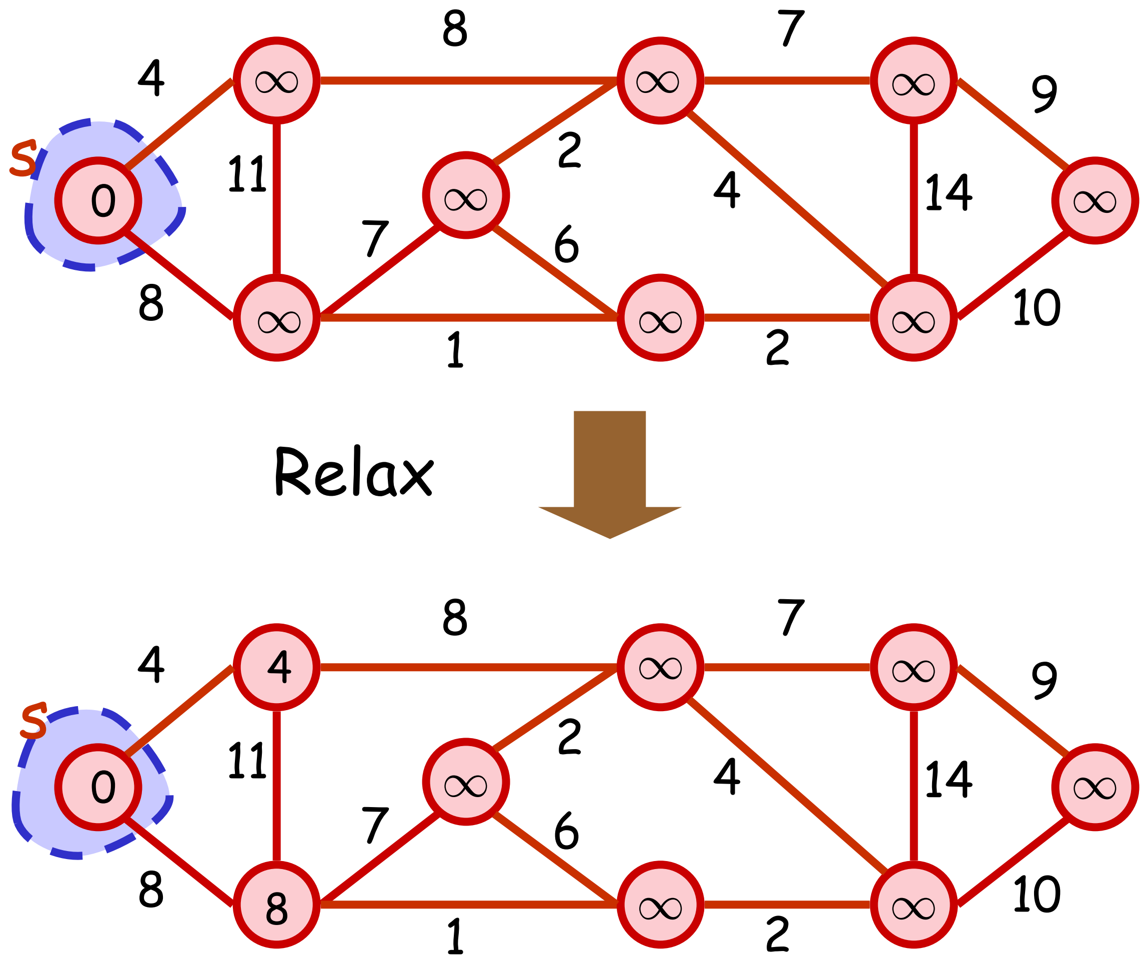
  when a vertex v can be reached from the source with a certain distance, we examine an outgoing edge, say (v,w), and check if we can improve w
- E.g.,



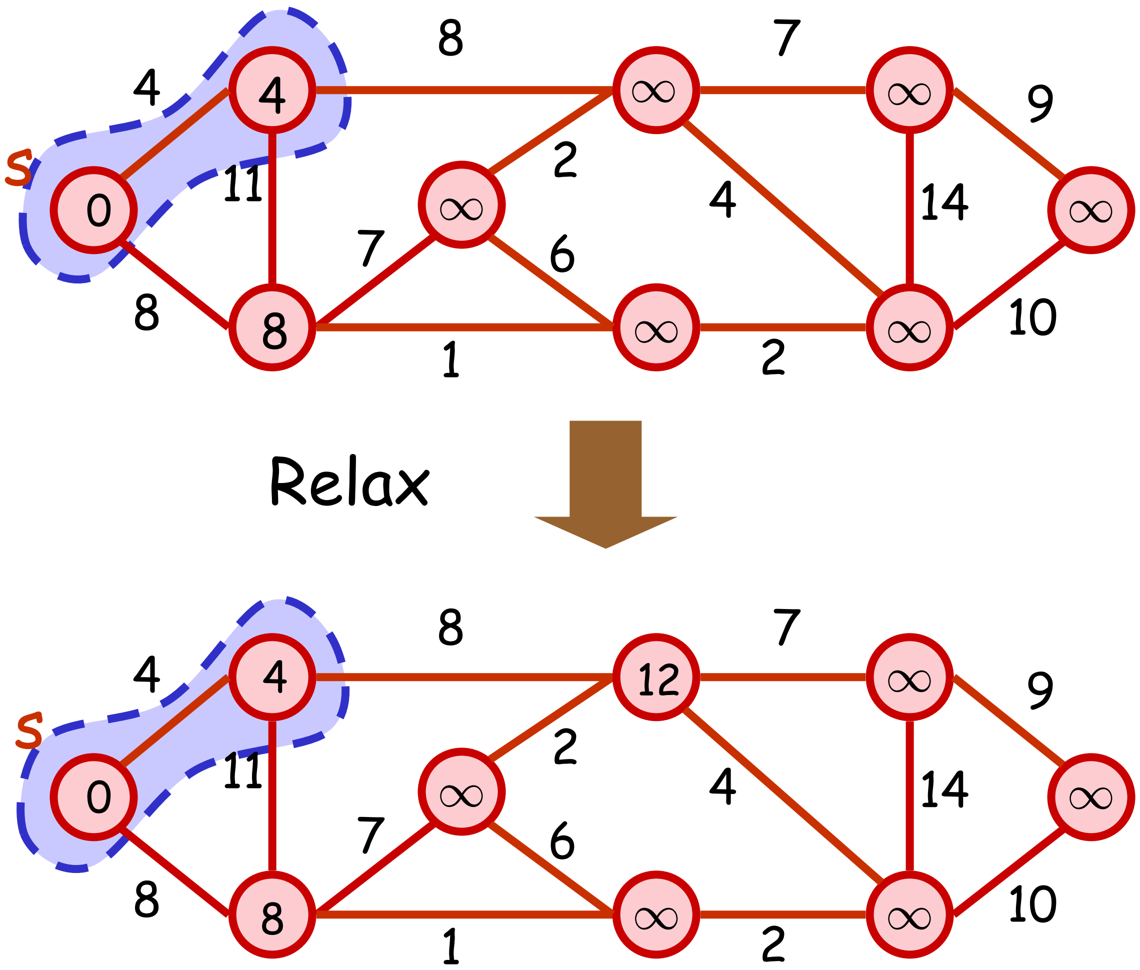
# Dijkstra's Algorithm

```
Dijkstra(G, s)
  For each vertex v,
      Mark v as unvisited, and set d(v) = ∞ ;
  Set d(s) = 0 ;
  while (there is unvisited vertex) {
     v = unvisited vertex with smallest d ;
     Visit v, and Relax all its outgoing edges;
  }
  return d ;
```

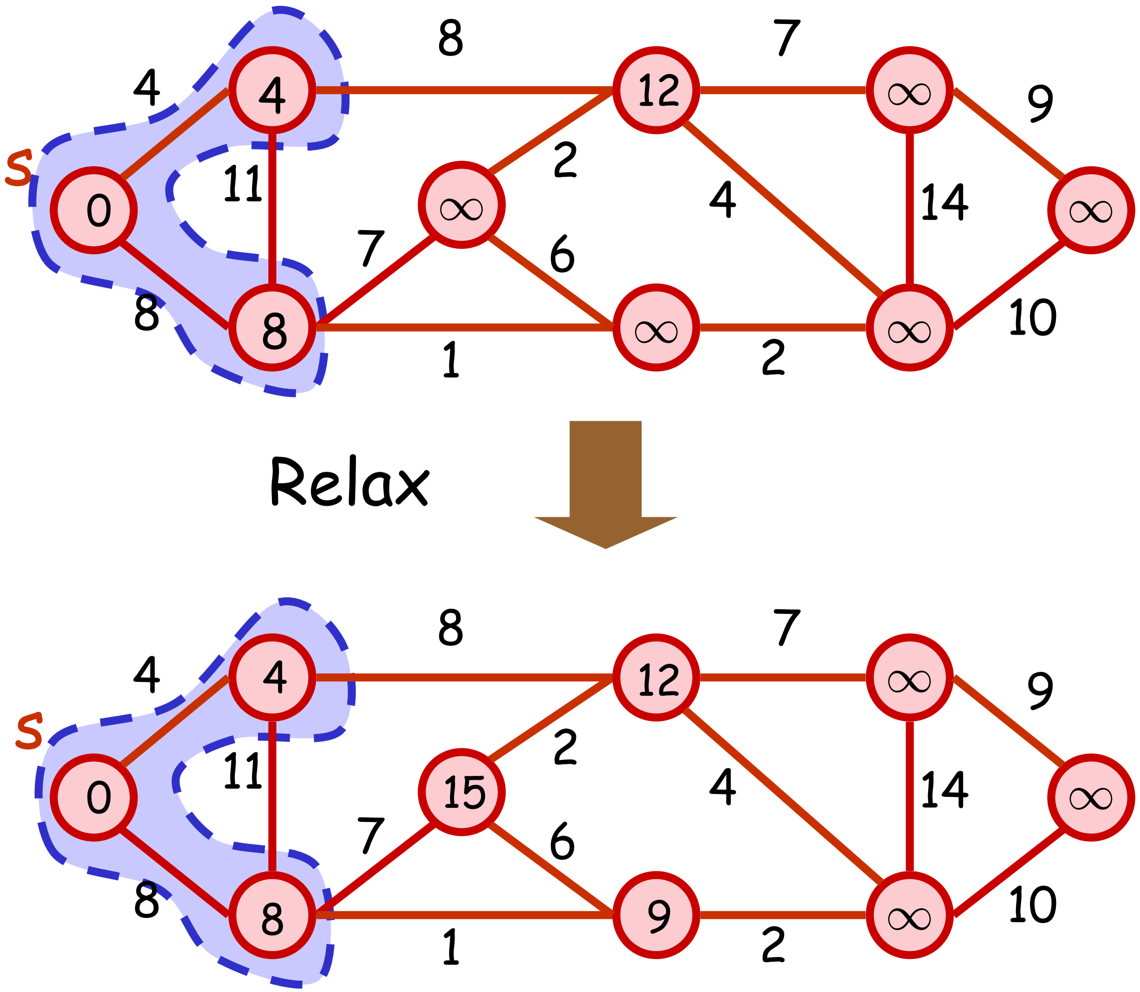
# Example



# Example



# Example



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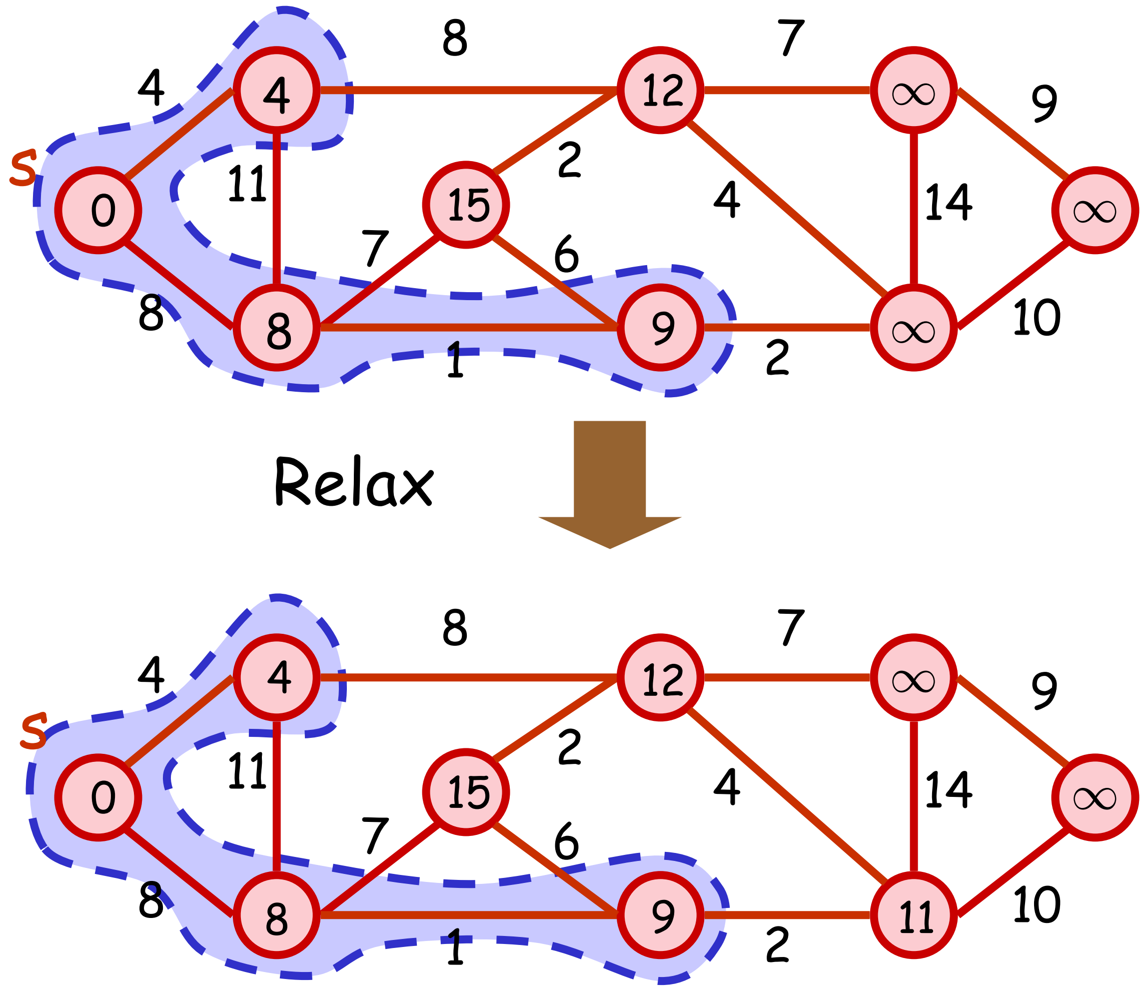
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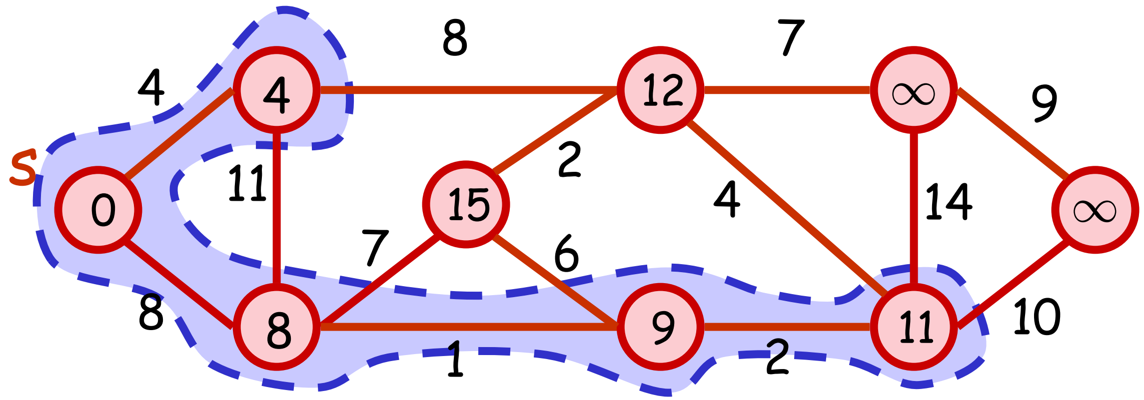
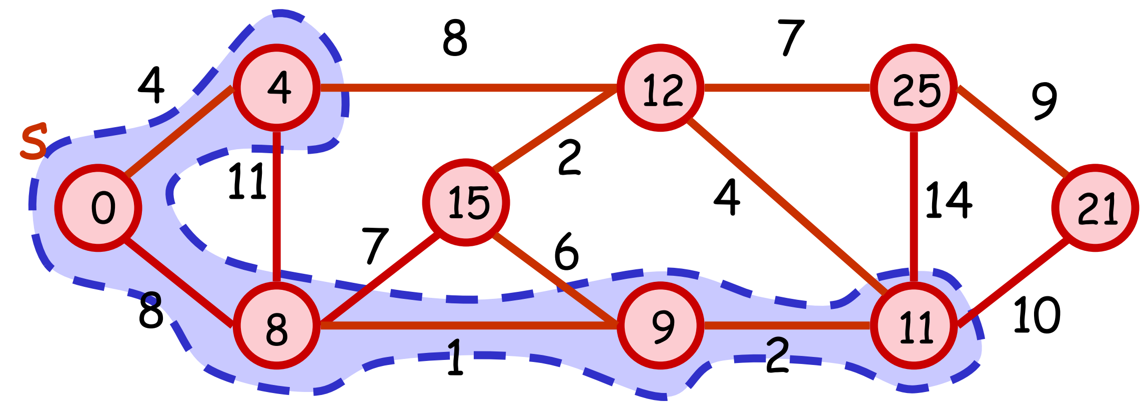
# Example




Relax



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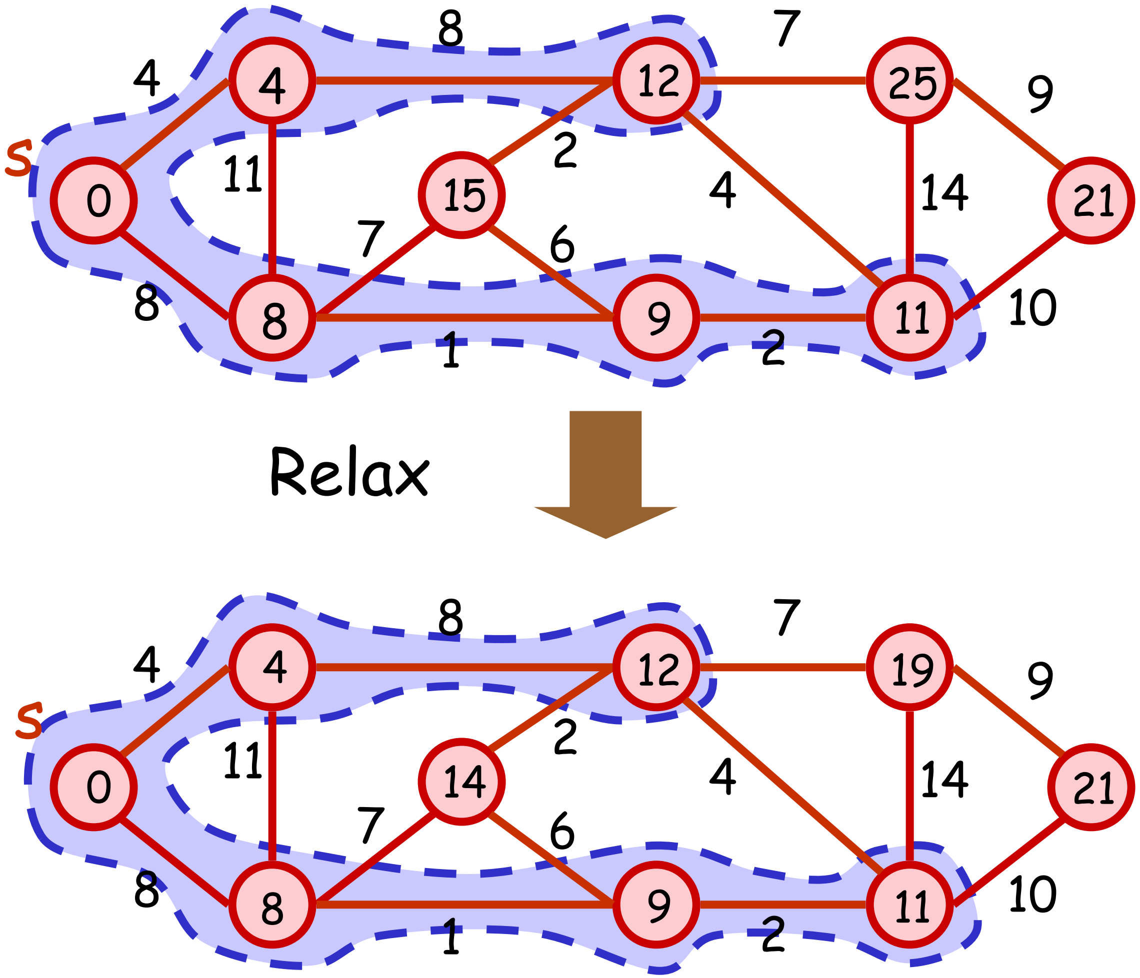
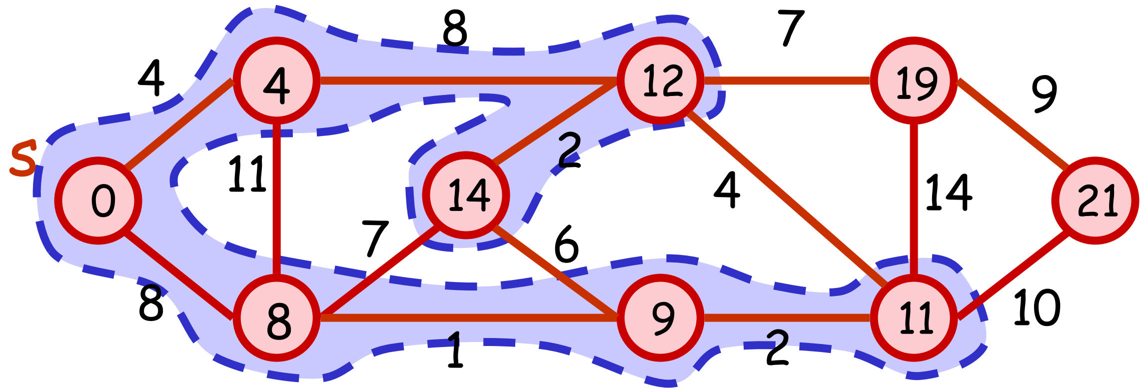
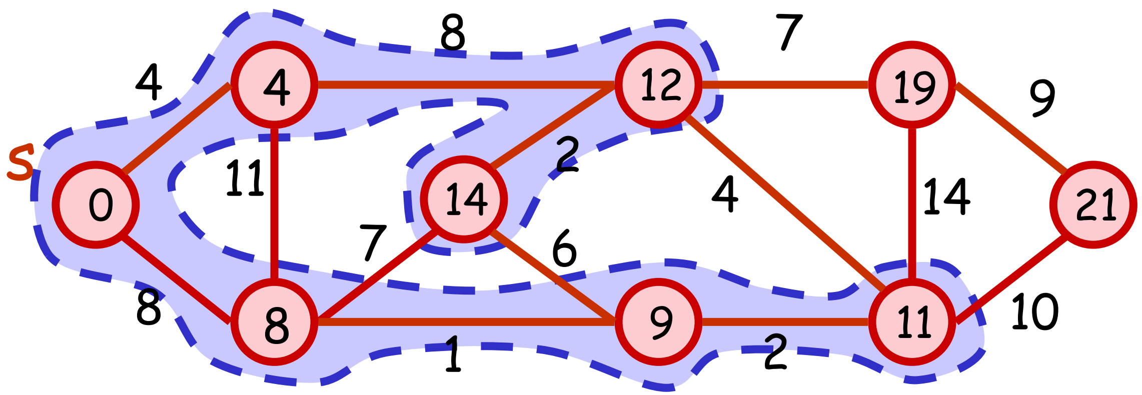


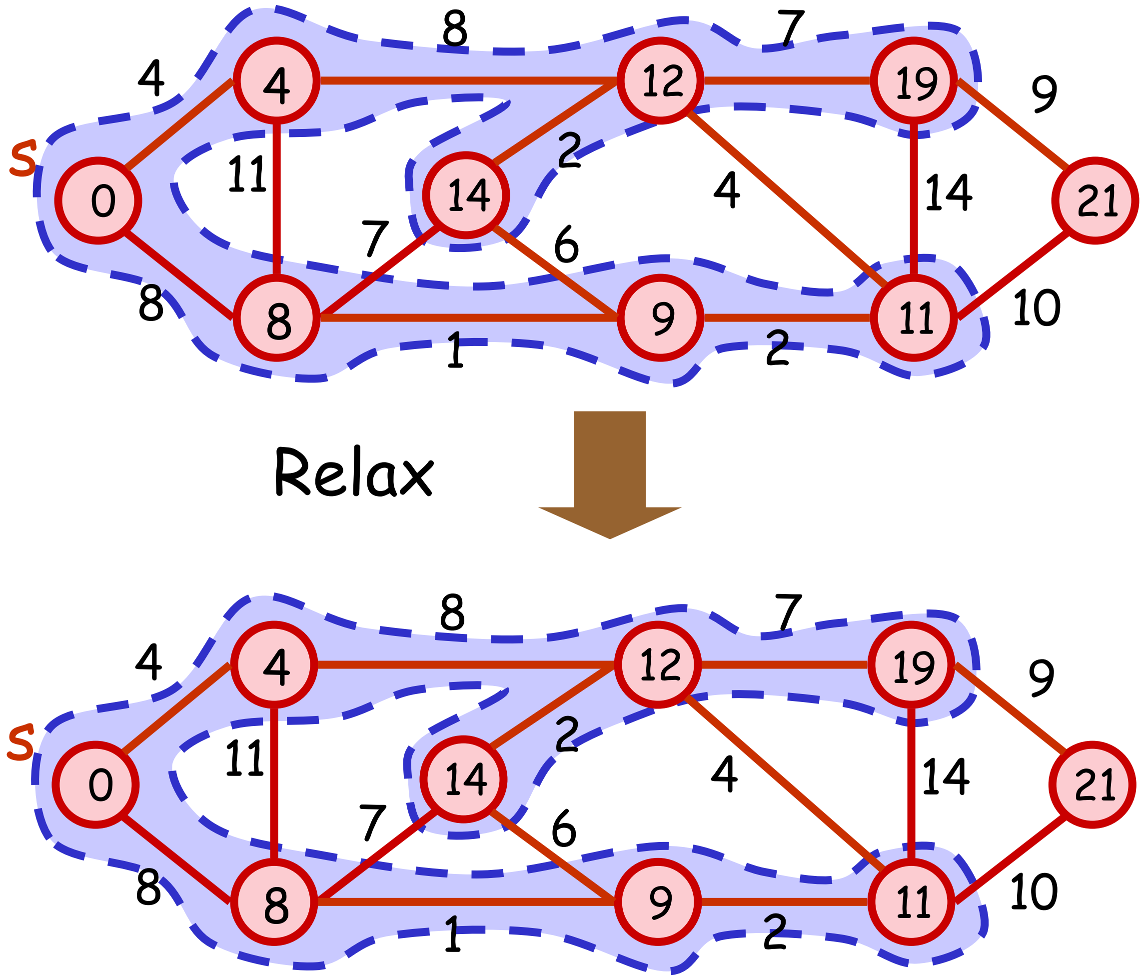
# Example




Relax



# Example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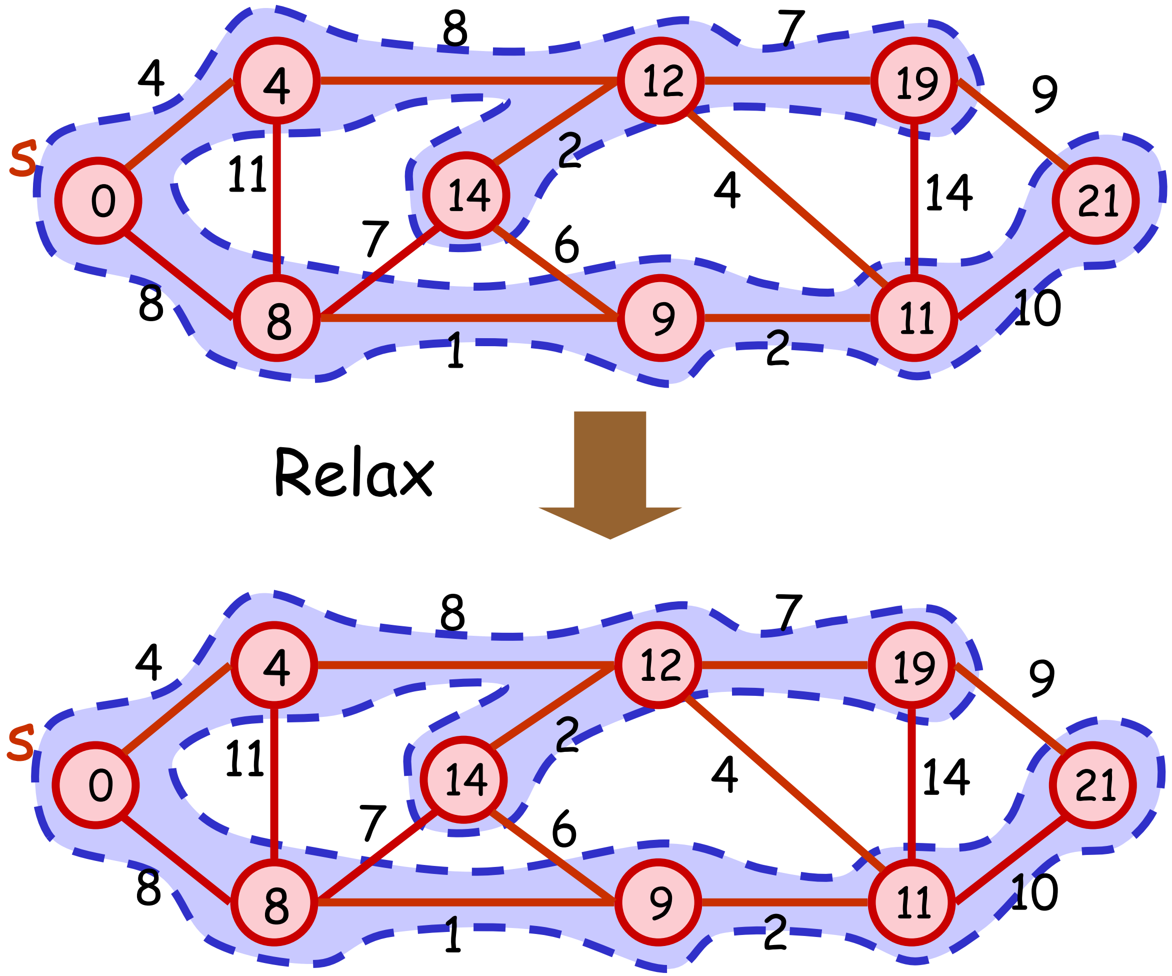
# Example



# Correctness

Theorem:

The $k^{th}$ vertex closest to the source $s$ is selected at the $k^{th}$ step inside the while loop of Dijkstra's algorithm

Also, by the time a vertex $v$ is selected, $d(v)$ will store the length of the shortest path from $s$ to $v$

How to prove ? (By induction)

# Proof

- Both statements are true for $k = 1$ ;
- Let $v_j$ = $j^{th}$ closest vertex from $s$
- Now, suppose both statements are true for $k = 1, 2, \ldots, r-1$
- Consider the $r^{th}$ closest vertex $v_r$
  - If there is no path from $s$ to $v_r$
    → $d(v_r) = \infty$ is never changed
  - Else, there must be a shortest path from $s$ to $v_r$ ; Let $v_t$ be the vertex immediately before $v_r$ in this path

# Proof (cont)

- Then, we have $t \leq r\text{-}1$ (why??)

→ $d(v_r)$ is set correctly once $v_t$ is selected, and the edge $(v_t, v_r)$ is relaxed (why??)

→ After that, $d(v_r)$ is fixed (why??)

→ $d(v_r)$ is correct when $v_r$ is selected ; also, $v_r$ must be selected at the $r^{th}$ step, because no unvisited nodes can have a smaller $d$ value at that time

Thus, the proof of inductive case completes

# Performance

- Dijkstra's algorithm is similar to Prim's
- By using Fibonacci Heap,
  - Relax ⇔ Decrease-Key
  - Pick vertex ⇔ Extract-Min
- Running Time:
  - $O(V)$ Insert/Extract-Min
  - At most $O(E)$ Decrease-Key

➔ Total Time: $O(E + V \log V)$

# Finding Shortest Path in DAG

We have a faster algorithm for DAG :

DAG-Shortest-Path($G$, $s$)

    Topological Sort $G$ ;

    For each v, set $d(v) = \infty$ ; Set $d(s) = 0$ ;

    for ($k$ = 1 to $|V|$) {

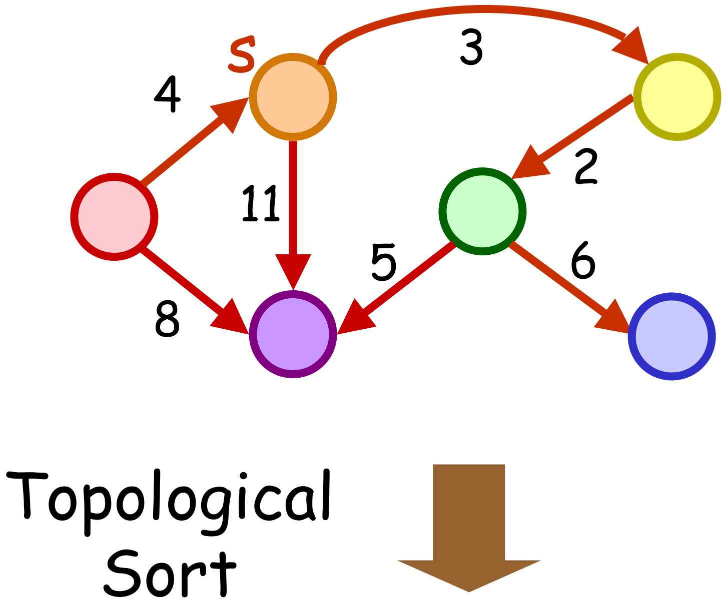
        v = $k^{th}$ vertex in topological order ;

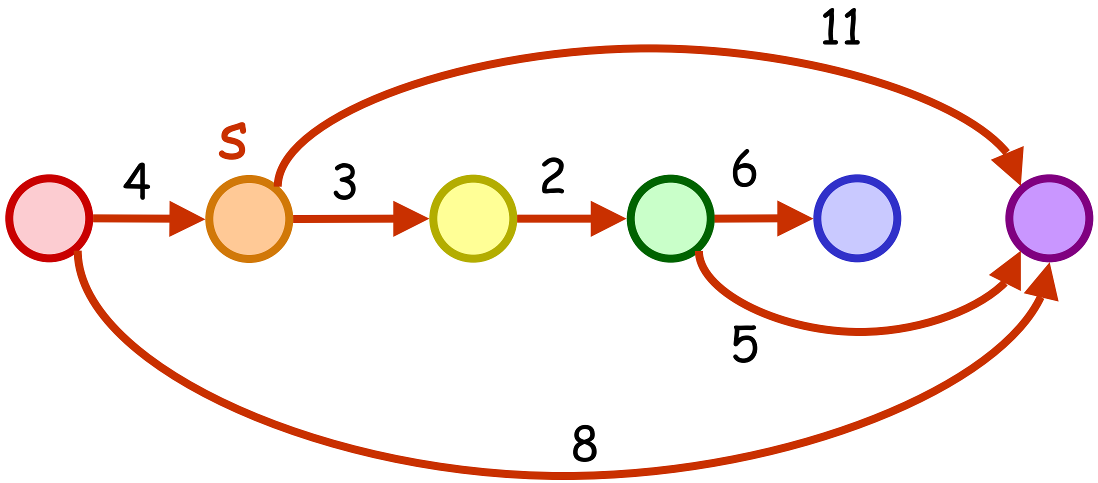
        Relax all outgoing edges of v ;

    }

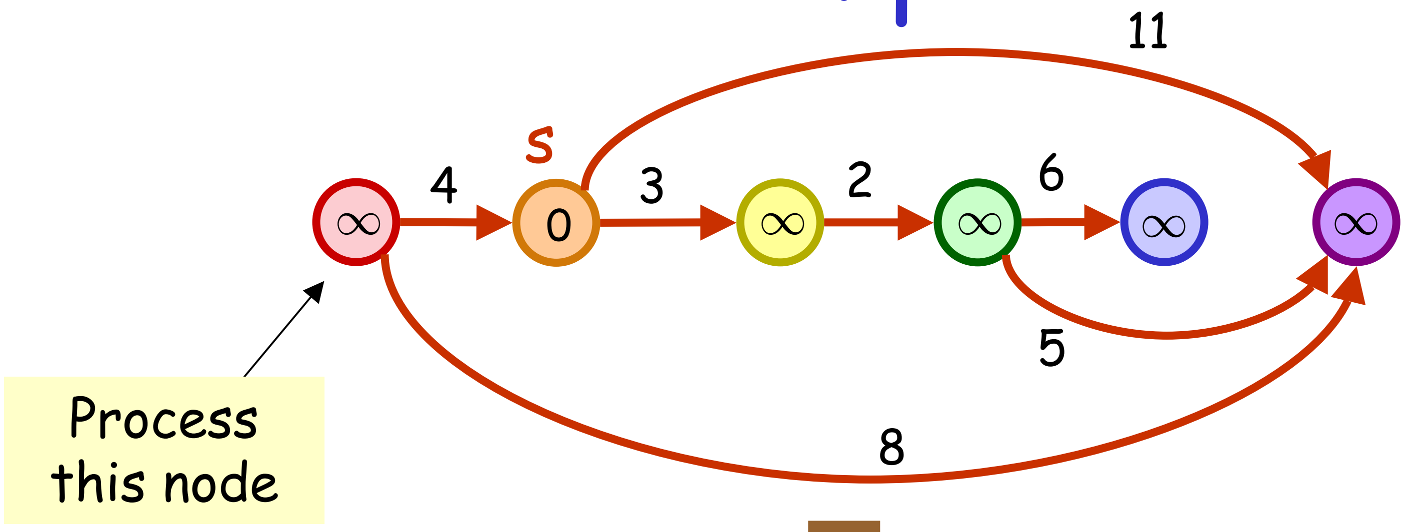
    return d ;

# Example

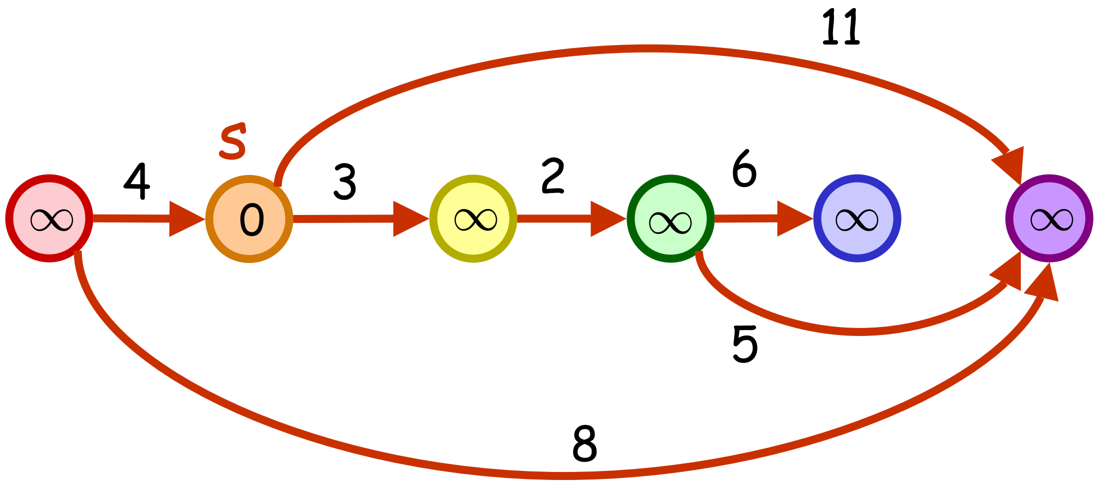


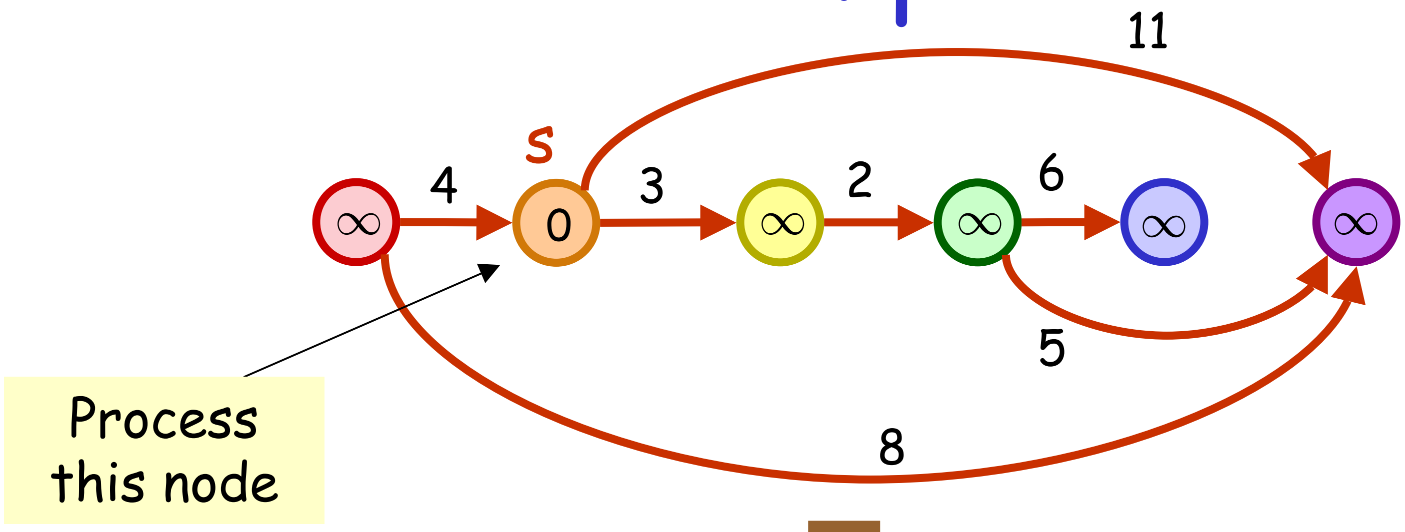
# Example



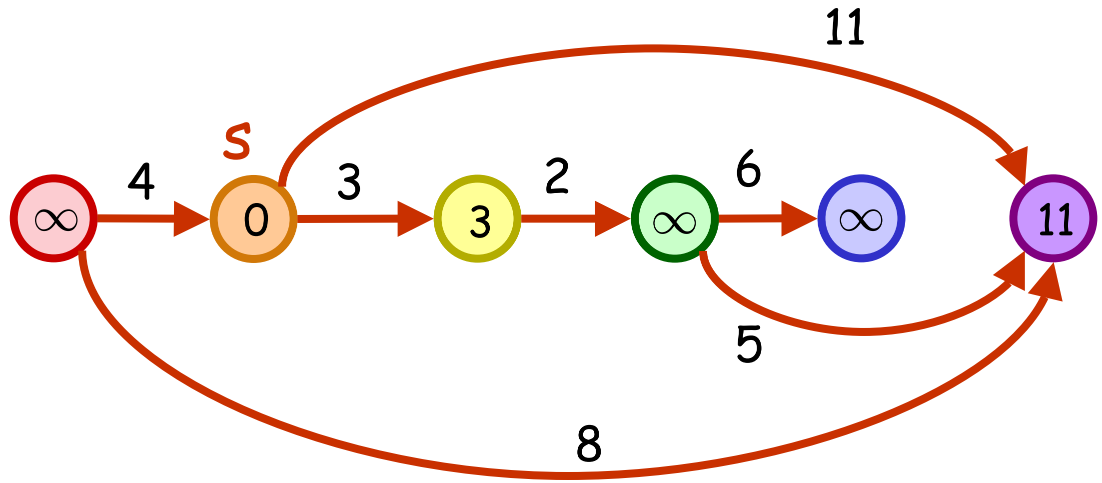

Relax



# Example




Relax



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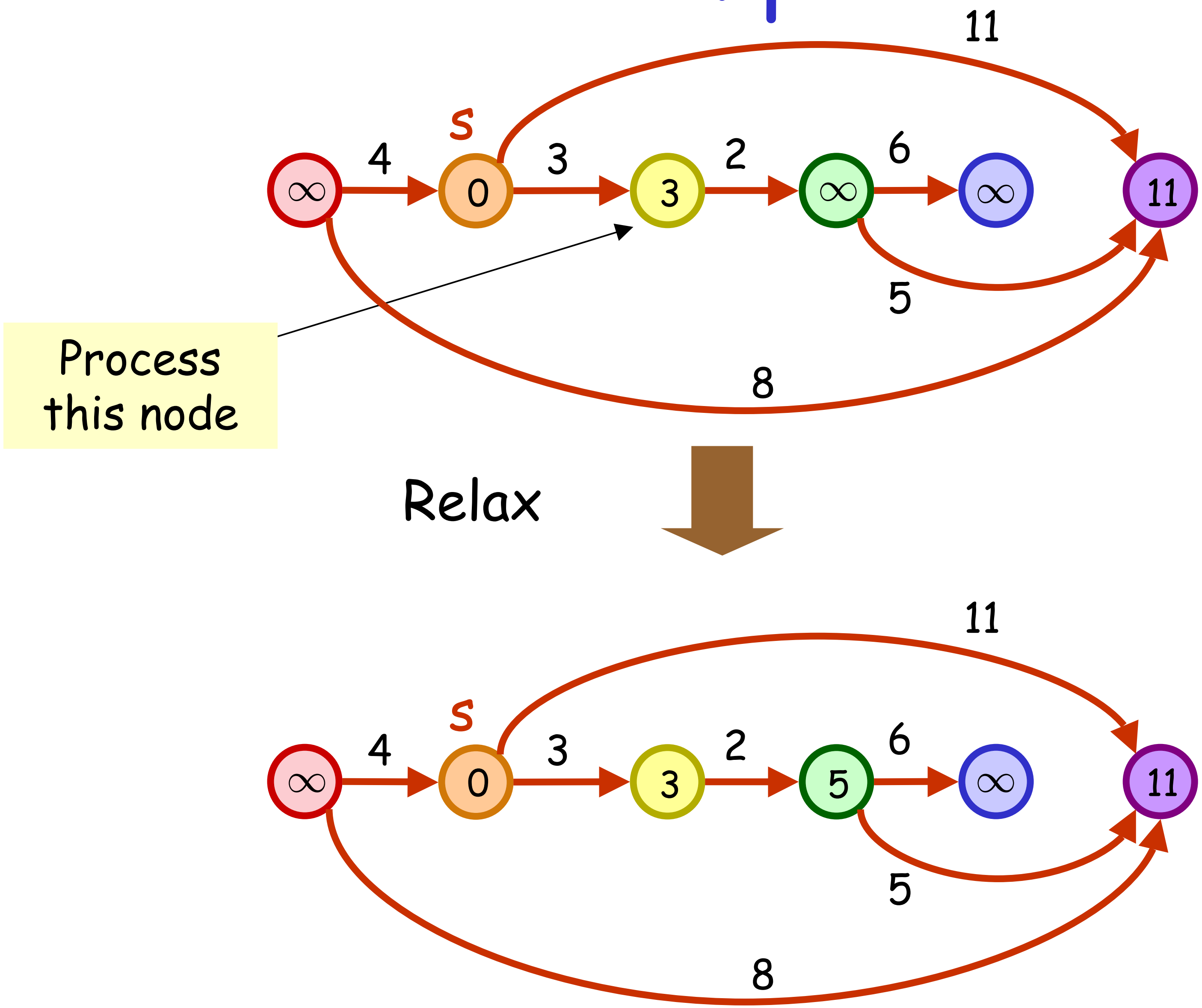

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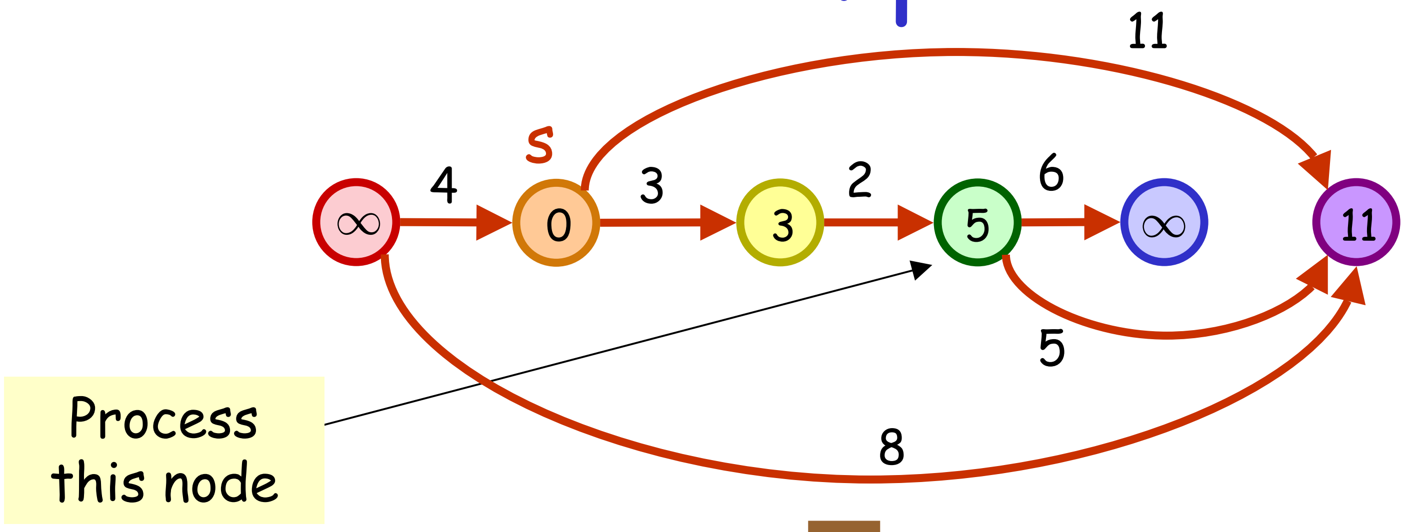


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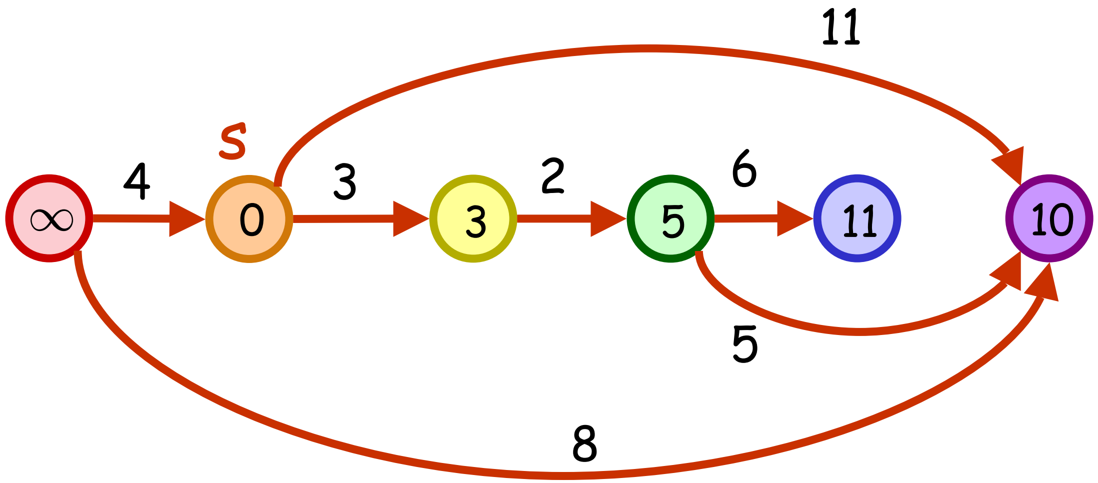

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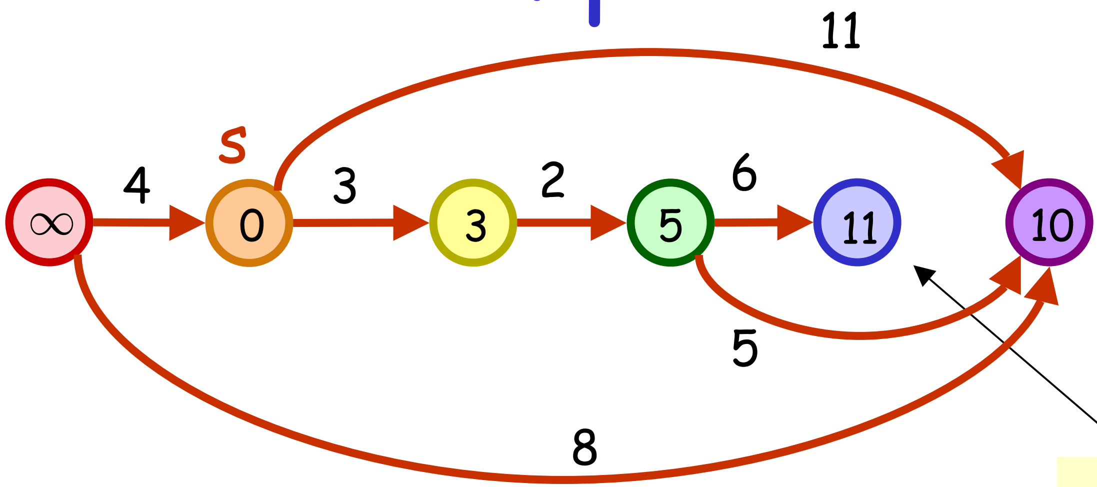
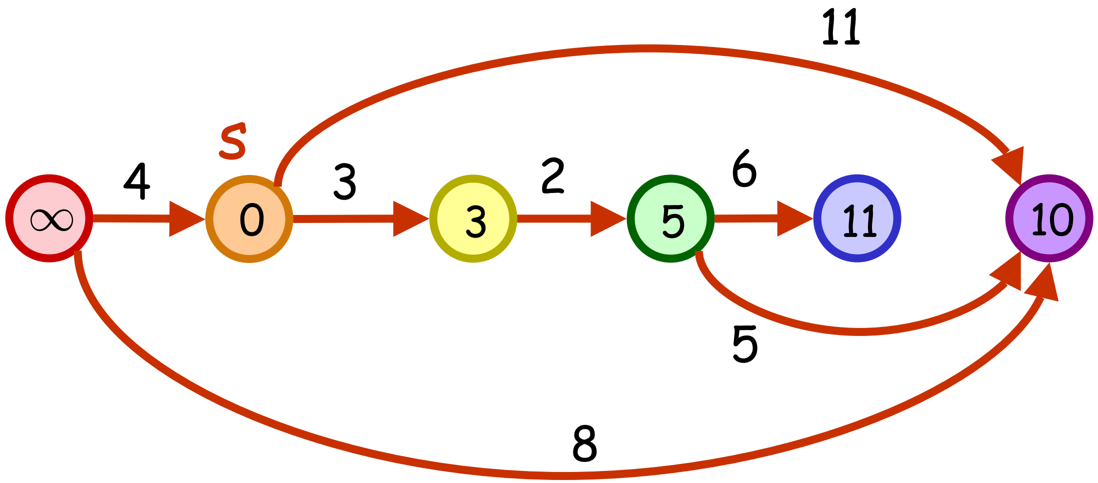
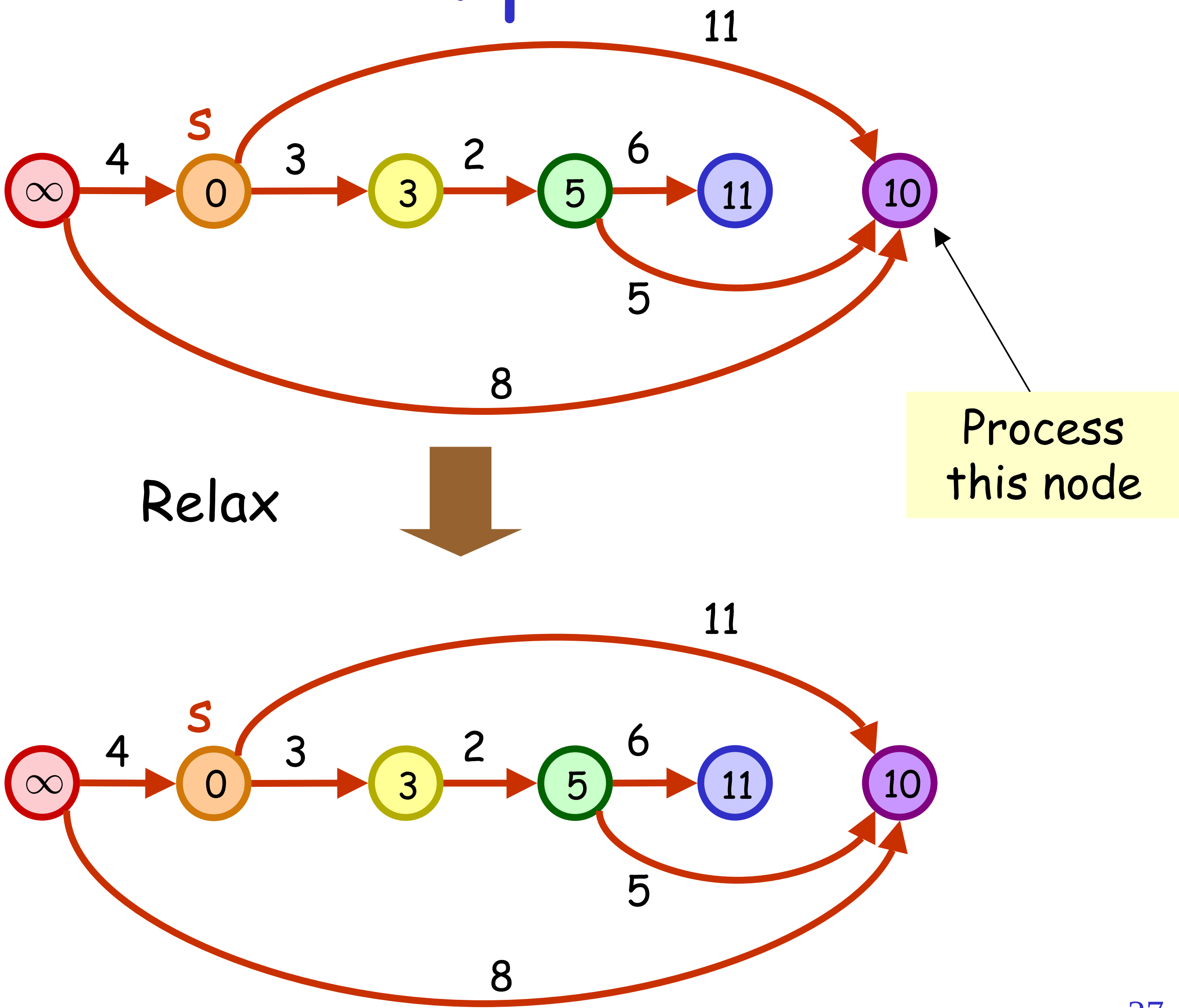



Process this node

Relax



# Example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Process this node
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# Correctness

Theorem:

By the time a vertex v is selected,
d(v) will store the length of the shortest path from s to v

How to prove ? (By induction)

# Proof

- Let $v_j$ = $j^{th}$ vertex in the topological order
- We will show that $d(v_k)$ is set correctly when $v_k$ is selected, for $k = 1,2, \ldots, |V|$
- When $k = 1$,

  $v_k = v_1$ = leftmost vertex

  If it is the source, $d(v_k) = 0$

  If it is not the source, $d(v_k) = \infty$

  → In both cases, $d(v_k)$ is correct (why?)

  → Base case is correct

# Proof (cont)

- Now, suppose the statement is true for $k = 1, 2, \ldots, r-1$
- Consider the vertex $v_r$
  - If there is no path from $s$ to $v_r$ ➔ $d(v_r) = \infty$ is never changed
  - Else, we shall use similar arguments as proving the correctness of Dijkstra's algorithm …

# Proof (cont)

- First, let $v_t$ be the vertex immediately before $v_r$ in the shortest path from $s$ to $v_r$

  → $t \le r-1$

  → $d(v_r)$ is set correctly once $v_t$ is selected, and the edge $(v_t, v_r)$ is relaxed

  → After that, $d(v_r)$ is fixed

  → $d(v_r)$ is correct when $v_r$ is selected

Thus, the proof of inductive case completes

# Performance

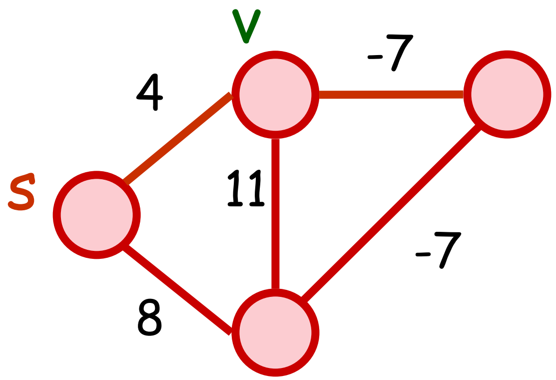
- DAG-Shortest-Path selects vertex sequentially according to topological order
  - no need to perform Extract-Min
- We can store the d values of the vertices in a single array ➔ Relax takes O(1) time
- Running Time:
  - Topological sort : O(V + E) time
  - O(V) select, O(E) Relax : O(V + E) time

➔ Total Time: O(V + E)

# Handling Negative Weight Edges

- When a graph has negative weight edges, shortest path may not be well-defined

E.g.,




What is the shortest path from s to v?

# Handling Negative Weight Edges

- The problem is due to the presence of a cycle *C*, reachable by the source, whose total weight is negative

  ➔ *C* is called a negative-weight cycle

- How to handle negative-weight edges ??

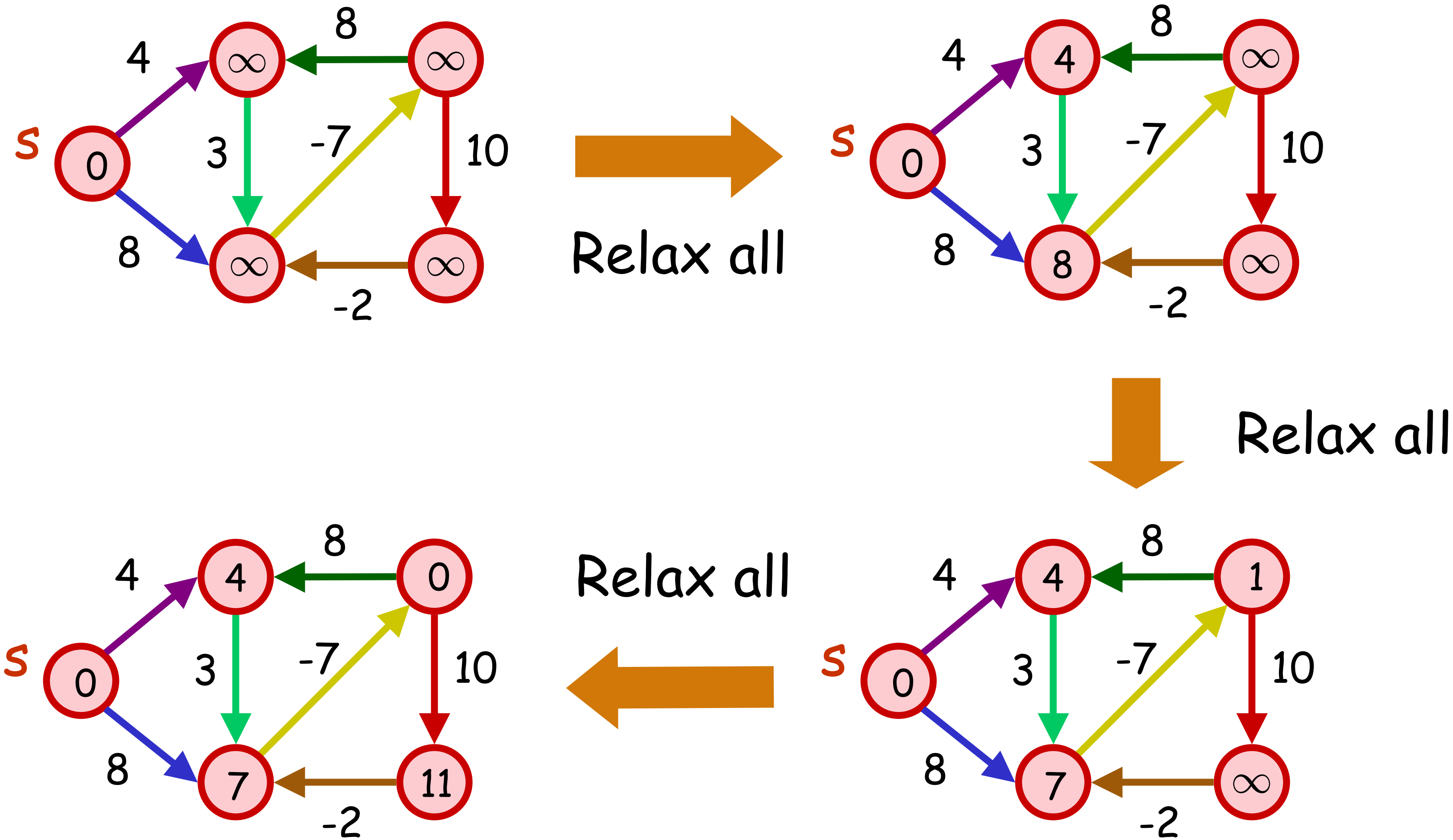
  ➔ if input graph is known to be a DAG, DAG-Shortest-Path is still correct

  ➔ For the general case, we can use Bellman-Ford algorithm

# Bellman-Ford Algorithm

```
Bellman-Ford(G, s)      // runs in O(VE) time
  For each v, set d(v) = ∞ ; Set d(s) = 0 ;
  for (k = 1 to |V|-1)
     Relax all edges in G in any order ;
  /* check if s reaches a neg-weight cycle */
  for each edge (u,v),
     if (d(v) > d(u) + weight(u,v))
          return "something wrong !!" ;
  return d ;
```

# Example 1



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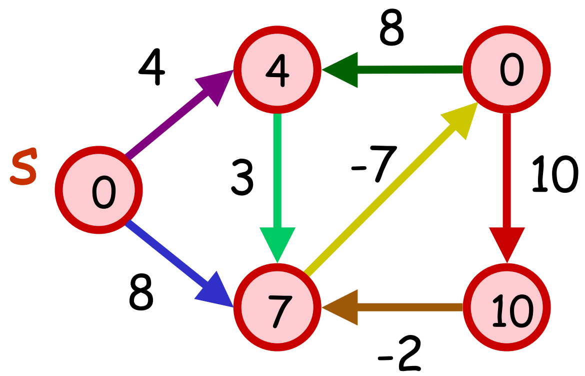
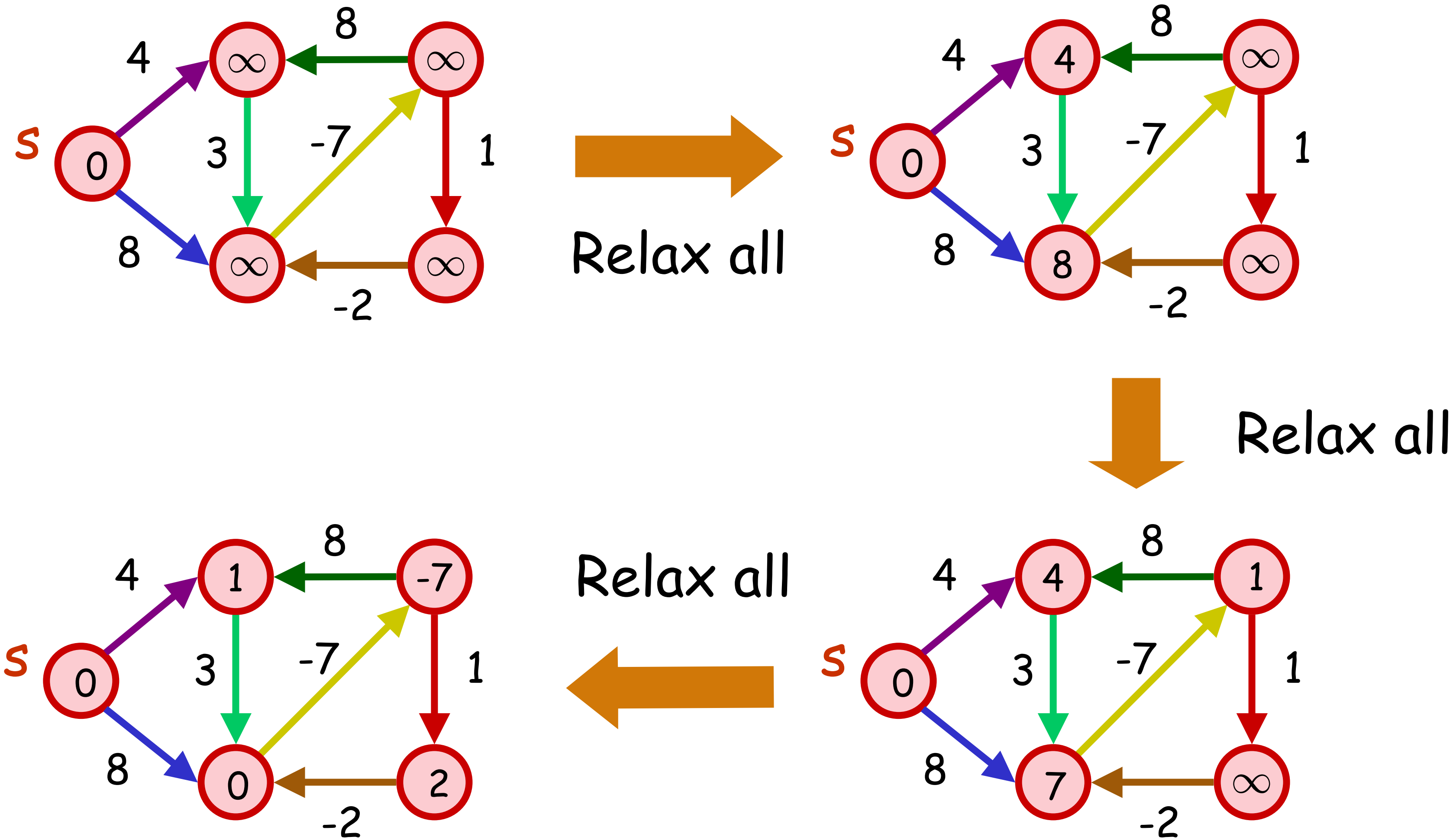
# Example 1

After the 4th Relax all



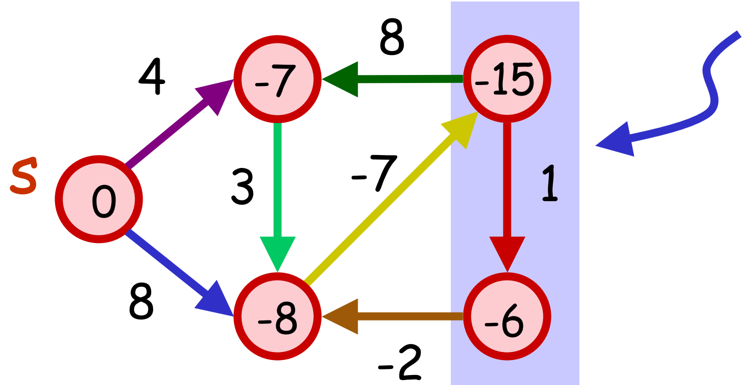

After checking, we found that there is nothing wrong ➔ distances are correct

# Example 2



# Example 2

After the 4th Relax all




This edge shows something must be wrong …

After checking, we found that something must be wrong ➔ distances are incorrect

# Correctness (Part 1)

Theorem:

If the graph has no negative-weight cycle, then for any vertex v with shortest path from s consists of k edges, Bellman-Ford sets d(v) to the correct value after the $k^{th}$ Relax all (for any ordering of edges in each Relax all )

How to prove ? (By induction)

# Corollary

Corollary: If there is no negative-weight cycle, then when Bellman-Ford terminates,

$$d(v) \le d(u) + \text{weight}(u,v)$$

for all edge (u,v)

Proof: By previous theorem, d(u) and d(v) are the length of shortest path from s to u and v, respectively. Thus, we must have

$$d(v) \le \text{length of any path from } s \text{ to } v$$

$$\Rightarrow d(v) \le d(u) + \text{weight}(u,v)$$

# “Something Wrong” Lemma

Lemma: If there is a negative-weight cycle, then when Bellman-Ford terminates,

$$d(v) > d(u) + weight(u,v)$$

for some edge (u,v)

How to prove ? (By contradiction)

# Proof

- Firstly, we know that there is a cycle
  $$C = (v_1, v_2, \ldots, v_k, v_1)$$
  whose total weight is negative

- That is, $\sum_{i=1 \text{ to } k} \text{weight}(v_i, v_{i+1}) < 0$

- Now, suppose on the contrary that
  $$d(v) \leq d(u) + \text{weight}(u,v)$$
  for all edge $(u,v)$ at termination

# Proof (cont)

- Can we obtain another bound for

$$\Sigma_{i = 1 \text{ to } k} \text{weight}(v_i, v_{i+1}) \ ?$$

- By rearranging, for all edge $(u,v)$

$$\text{weight}(u,v) \geq d(v) - d(u)$$

$\Rightarrow$ $\Sigma_{i = 1 \text{ to } k} \text{weight}(v_i, v_{i+1})$

$\geq \Sigma_{i = 1 \text{ to } k} (d(v_i) - d(v_{i+1})) = 0$ (why?)

$\Rightarrow$ Contradiction occurs !!

# Correctness (Part 2)

- Combining the previous corollary and lemma, we have:

**Theorem:**

There is a negative-weight cycle in the input graph if and only if when Bellman-Ford terminates,

$$d(v) > d(u) + weight(u,v)$$

for some edge $(u,v)$