CS4311
Design and Analysis of Algorithms

Lecture 25:
Elementary Graph Algorithms IV

About this lecture

- Review of Strongly Connected Components (SCC) in a directed graph

- Finding all SCC
  (i.e., decompose a directed graph into SCC)

Mutually Reachable

- Let G be a directed graph
- Let u and v be two vertices in G

Definition: If u can reach v (by a path) and v can reach u (by a path), then we say u and v are mutually reachable

We shall use the notation $u \leftrightarrow v$ to indicate u and v are mutually reachable

Also, we assume $u \leftrightarrow u$ for any node u

Mutually Reachable

Theorem: $\leftrightarrow$ is an equivalence relation

Proof:

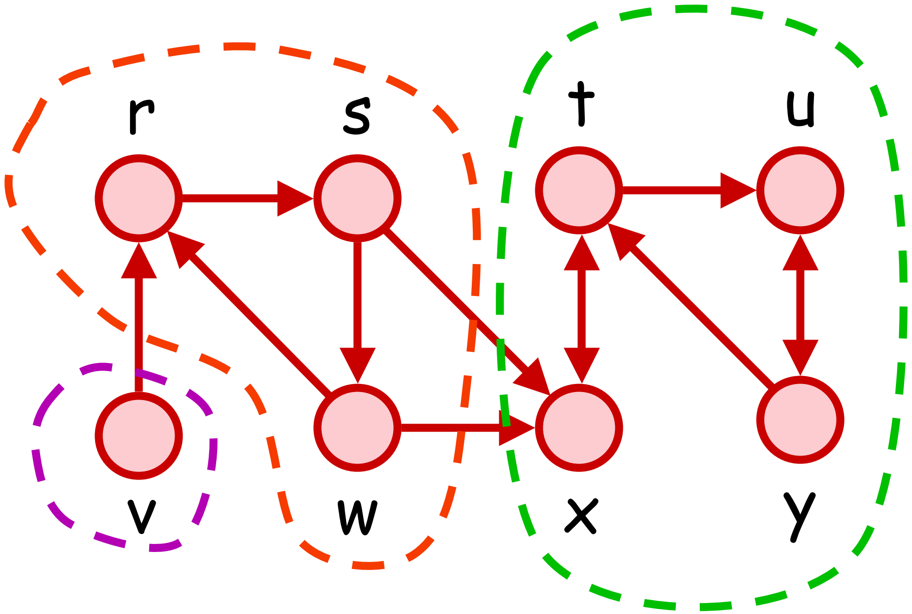
- By assumption $u \leftrightarrow u$, so $\leftrightarrow$ is reflexive
- If $u \leftrightarrow v$, then $v \leftrightarrow u$, so $\leftrightarrow$ is symmetric
- Also, if $u \leftrightarrow v$ and $v \leftrightarrow w$, then $u \leftrightarrow w$, so $\leftrightarrow$ is transitive

Thus, $\leftrightarrow$ is an equivalence relation

Strongly Connected Components

- Thus, we can partition V based on $\leftrightarrow$
- Let $V_1, V_2, \ldots, V_k$ denote the partition
- Each V_i is called a strongly connected component (SCC) of G

E.g.,



Property of SCC

- Let $G = (V, E)$ be a directed graph
- Let G^T be a graph obtained from G by reversing the direction of every edge in G

  ➔ Adjacency matrix of G^T

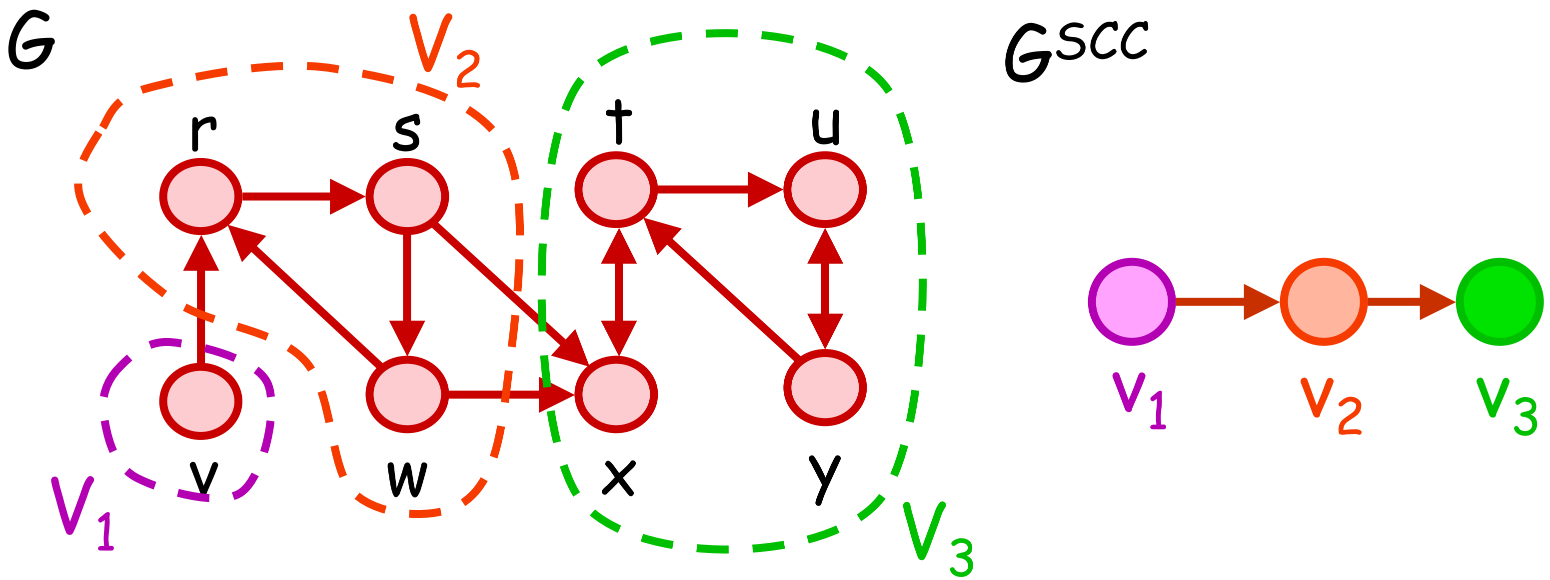
  = transpose of adjacency matrix of G

Theorem:

G and G^T has the same set of SCC's

Property of SCC

- Let $V_1, V_2, \ldots, V_k$ denote SCC of a graph G
- Let G^{SCC} be a simple graph obtained by contracting each V_i into a single vertex v_i
  - We call G^{SCC} the component graph of G



Property of G^{SCC}

Theorem: G^{SCC} is acyclic

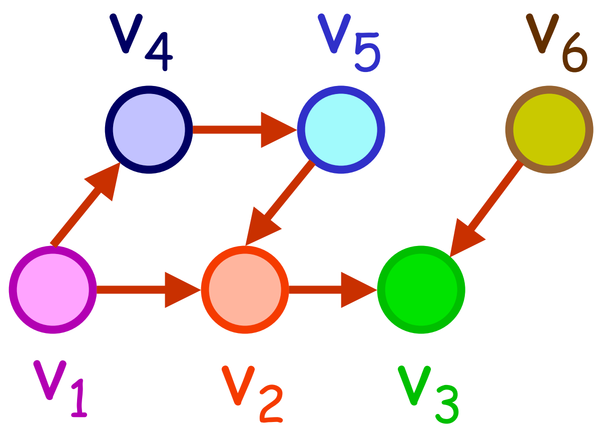
Proof: (By contradiction)

If G^{SCC} has a cycle, then there are some vertices v_i and v_j with $v_i \leftrightarrow v_j$

By definition, v_i and v_j correspond to two distinct SCC V_i and V_j in G. However, we see that any pair of vertices in V_i and V_j are mutually reachable → contradiction

Property of G^{SCC}

- Suppose the DAG (directed acyclic graph) on the right side is the G^{SCC} of some graph G



- Now, suppose we perform DFS on G
  - let u = node with largest finishing time

Question: Which SCC can u be located ?

Property of G^{SCC}

Lemma:

Consider any graph G.

Let G^{SCC} be its component graph.

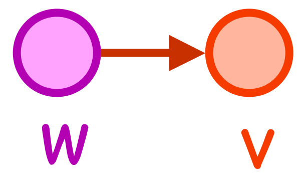
Suppose v is a vertex in G^{SCC} with at least one incoming edge.

Then, the node finishing last in any DFS of G cannot be a vertex of the SCC corresponding to v

Proof

- Let SCC(v) = SCC corresponding to v
- Since v has incoming edge, there exists w such that (w,v) is an edge in G^{SCC}

Inside G^{SCC}



- In the next two slides, we shall show that some node in SCC(w) must finish later than any node in SCC(v)
  - Consequently, u cannot be in SCC(v)

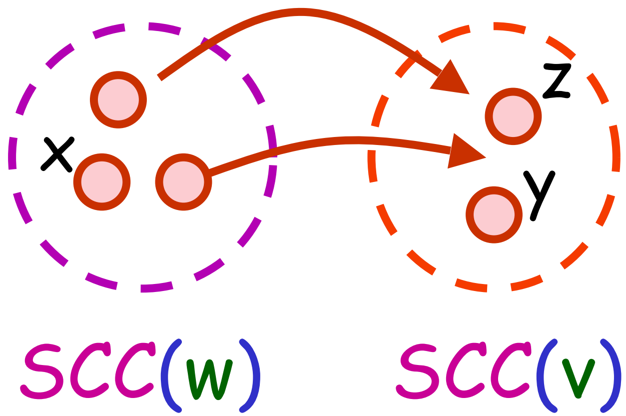
Proof

Let x = 1st node in SCC(w) discovered by DFS

Let y = 1st node in SCC(v) discovered by DFS

Let z = last node in SCC(v) finished by DFS

Inside G



// Note: z may be the same as y

By white-path theorem, we must have

$$d(y) \leq d(z) < f(z) \leq f(y)$$

Proof

If $d(x) < d(y)$

- then y becomes x's descendant (by white-path)

  $\rightarrow f(z) \leq f(y) < f(x)$

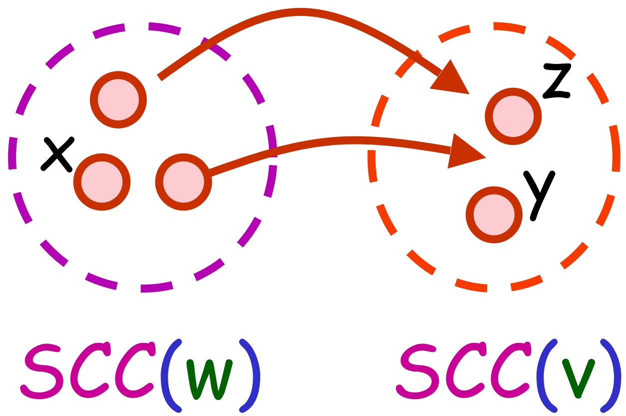
If $d(y) < d(x)$

- since x cannot be y's descendant (otherwise, they are in the same SCC)

  $\rightarrow d(y) < f(y) < d(x) < f(x)$

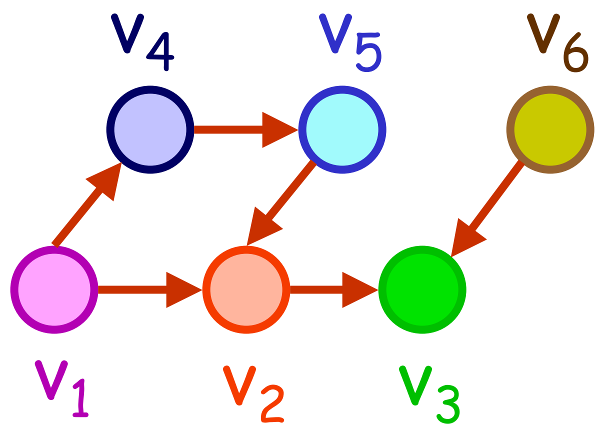
  $\rightarrow f(z) \leq f(y) < f(x)$

Inside G

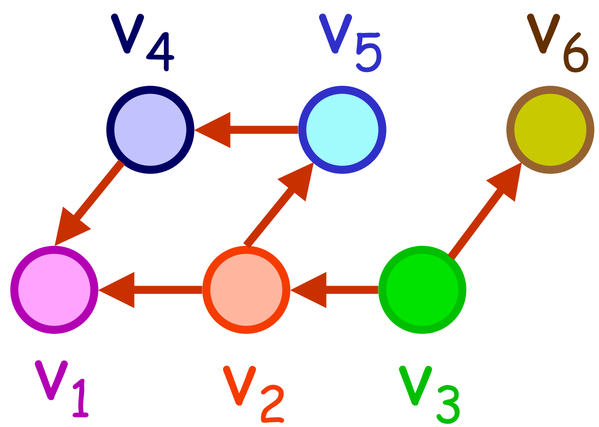


Finding SCC

- So, we know that u (last finished node of G) must be in an SCC with no incoming edges
- Let us reverse edge directions, and start DFS on G^T from u



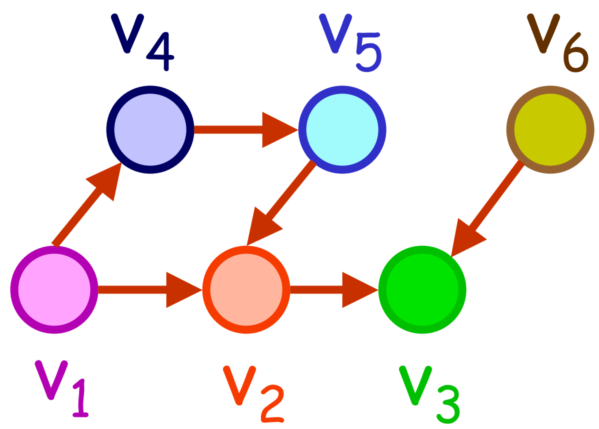
Question: Who will be u's descendants ??



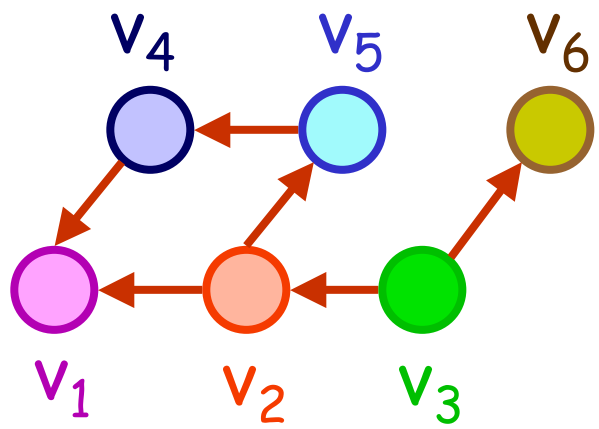
New G^{SCC}

Finding SCC

- Note that nodes in the SCC containing u cannot connect to nodes in other SCCs in G^T



- By white-path theorem, the descendants of u in G^T must be exactly those nodes in the same SCC as u



New G^{SCC}

Finding SCC

- Once DFS on u inside G^T has finished, all nodes in the same SCC as u are finished
  - ➔ Any subsequent DFS in G^T will be made as if this SCC was removed from G^T
- Now, let u' be the remaining node in G^T whose finishing time (in DFS in G) is latest
  - Where can u' be located?
  - Who will be the descendents of u' if we perform DFS in G^T now?

- Our observations lead to the following algorithm for finding all SCCs of G :

```
Finding-all-SCC(G) {
    1. Perform DFS on G ;
    2. Construct G^T ;
    3. while (some node in G^T is undiscovered)
       {  u = undiscovered node with latest
                finishing time** ;
          Perform DFS on G^T from u ;
       }  // nodes in the DFS tree from u forms an SCC
}  // ** Finishing times always refer to Step 1's DFS
```

Correctness & Performance

- The correctness of the algorithm can be proven by induction

  (Hint: Show that at each sub-search in Step 3, u is chosen from an SCC which has no outgoing edges to any nodes in an "unvisited" SCC of G^T.

  ➔ By white-path theorem, exactly all nodes in the same SCC become u's descendants)

- Running Time: $O(|V|+|E|)$ (why?)