CS4311
Design and Analysis of Algorithms

Lecture 24:
Elementary Graph Algorithms III

About this lecture

- Depth First Search
  - Classification of Tree Edges

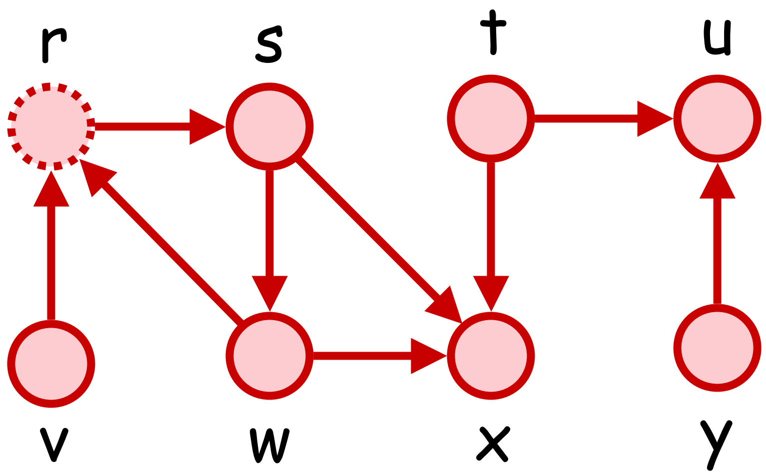
- Topological Sort

Classification of Tree Edges

- After a DFS process, we can classify the edges of a graph into four types :
  1. Tree : Edges in the DFS forest
  2. Back : From descendant to ancestor when explored (include self loop)
  3. Forward : From ancestor to descendant when explored (exclude tree edge)
  4. Cross : Others (no ancestor-descendant relation)

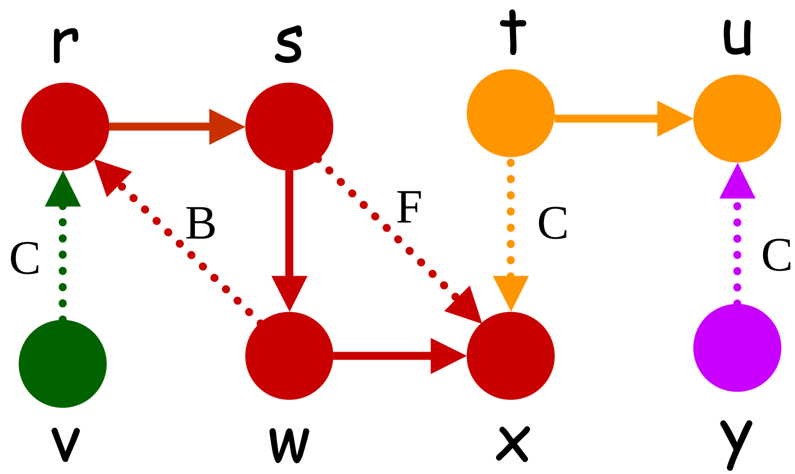
Example

Suppose the input graph is directed



Example

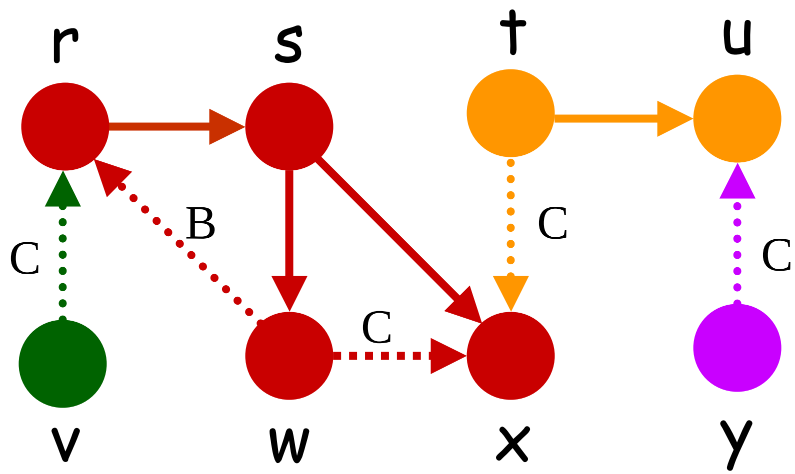
Suppose this is the DFS forest obtained



Can you classify the type of each edge ?

Example

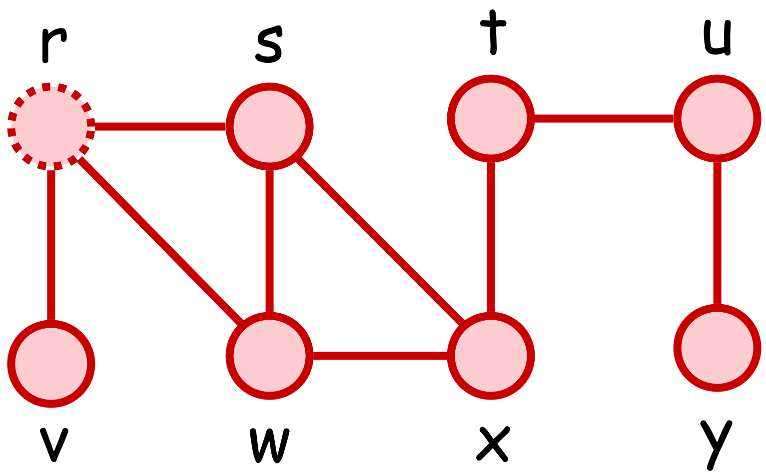
Suppose the DFS forest is different now …



Can you classify the type of each edge ?

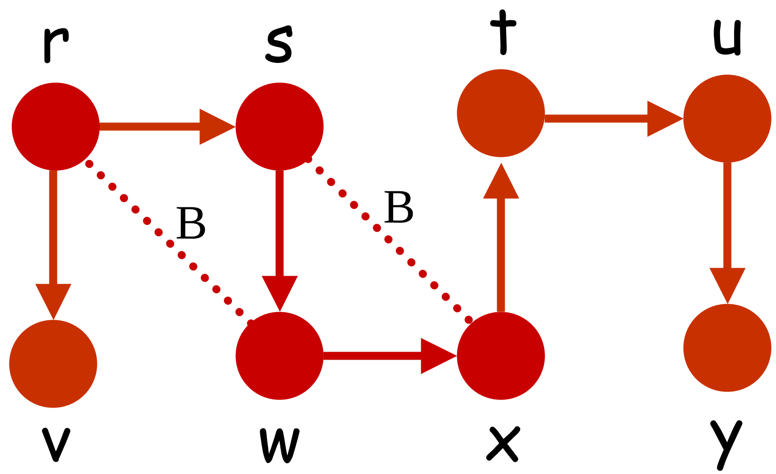
Example

Suppose the input graph is undirected



Example

Suppose this is the DFS forest obtained



Can you classify the type of each edge ?

Edges in Undirected Graph

Theorem:

After DFS of an undirected graph, every edge is either a tree edge or a back edge

Proof: Let (u,v) be an edge. Suppose u is discovered first. Then, v will become u's descendent (white-path) so that $f(v) < f(u)$

- If u discovers v ➔ (u,v) is tree edge
- Else, (u,v) is explored after v discovered

  Then, (u,v) must be explored from v because $f(v) < f(u)$ ➔ (u,v) is back edge

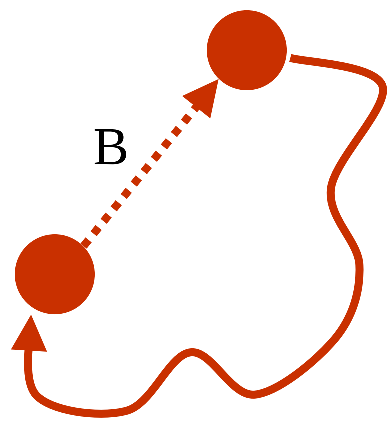
Cycles in Directed Graph

Theorem: For any DFS on a directed graph G, there is a back edge ⇔ G has a cycle

Proof:

(⇒) If there is a back edge (u,v), it implies there is a path from v to u.

Thus, this back edge completes a cycle



Proof

($\Leftarrow$) If G has a cycle C, let

v = first vertex discovered in DFS

(u,v) = v's preceding edge in C

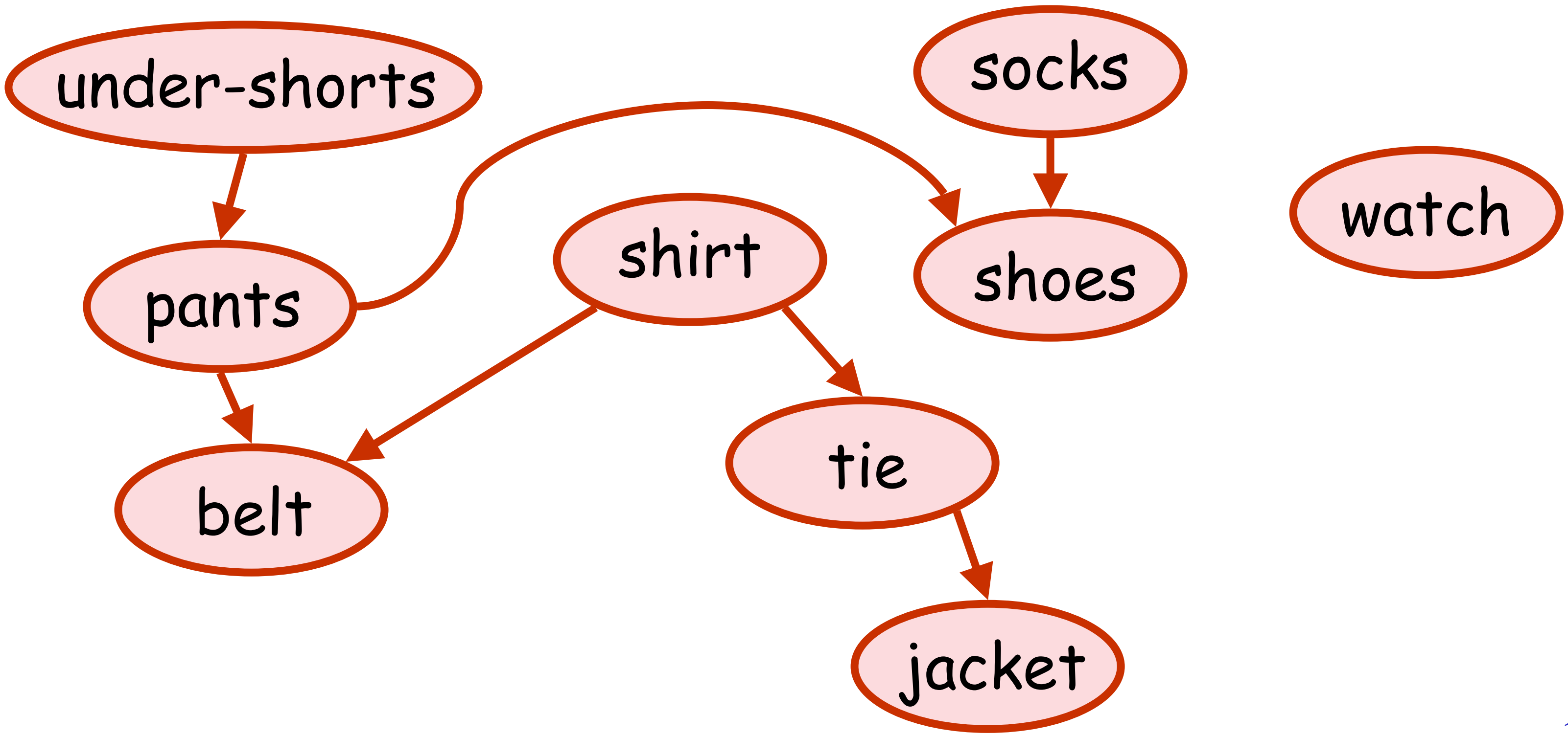
Thus, when v is discovered, all nodes in C are still undiscovered (white)

$\rightarrow$ v is ancestor of u in DFS forest (why?)

$\rightarrow$ (u,v) becomes a back edge

Topological Sort

- Directed graph can be used to indicate precedence among a set of events
- E.g., a possible precedence is dressing



Topological Sort

- The previous directed graph is also called a precedence graph

Question: Given a precedence graph G, can we order the events such that if (u,v) is in G (i.e. u should complete before v) then u appears before v in the ordering ?

We call this problem topological sorting of G

Topological Sort

Fact: If G contains a cycle, then it is impossible to find a desired ordering

(Prove by contradiction)

- However, if G is acyclic (not contains any cycle) we show that the algorithm in next slide always find one of the desired ordering

Topological Sort

Topological-Sort(G)

{

1. Call DFS on G
2. If G contains a back edge, abort ;
3. Else, output vertices in decreasing order of their finishing times ;

}

Why is the algorithm correct?

Topological Sort

Theorem:

If G is acyclic, the previous algorithm produces a topological sort of G

Proof: Let (u,v) be an edge in G. We shall show that $f(u) > f(v)$ so that u appears before v in the output ordering

Recall G is acyclic, there is no back edges.

There are two main cases …

Proof

- Case 1: (u,v) is a tree or forward edge

  ➔ u is an ancestor of v

  ➔ $d(u) < d(v) < f(v) < f(u)$ (why??)

- Case 2: (u,v) is a cross edge

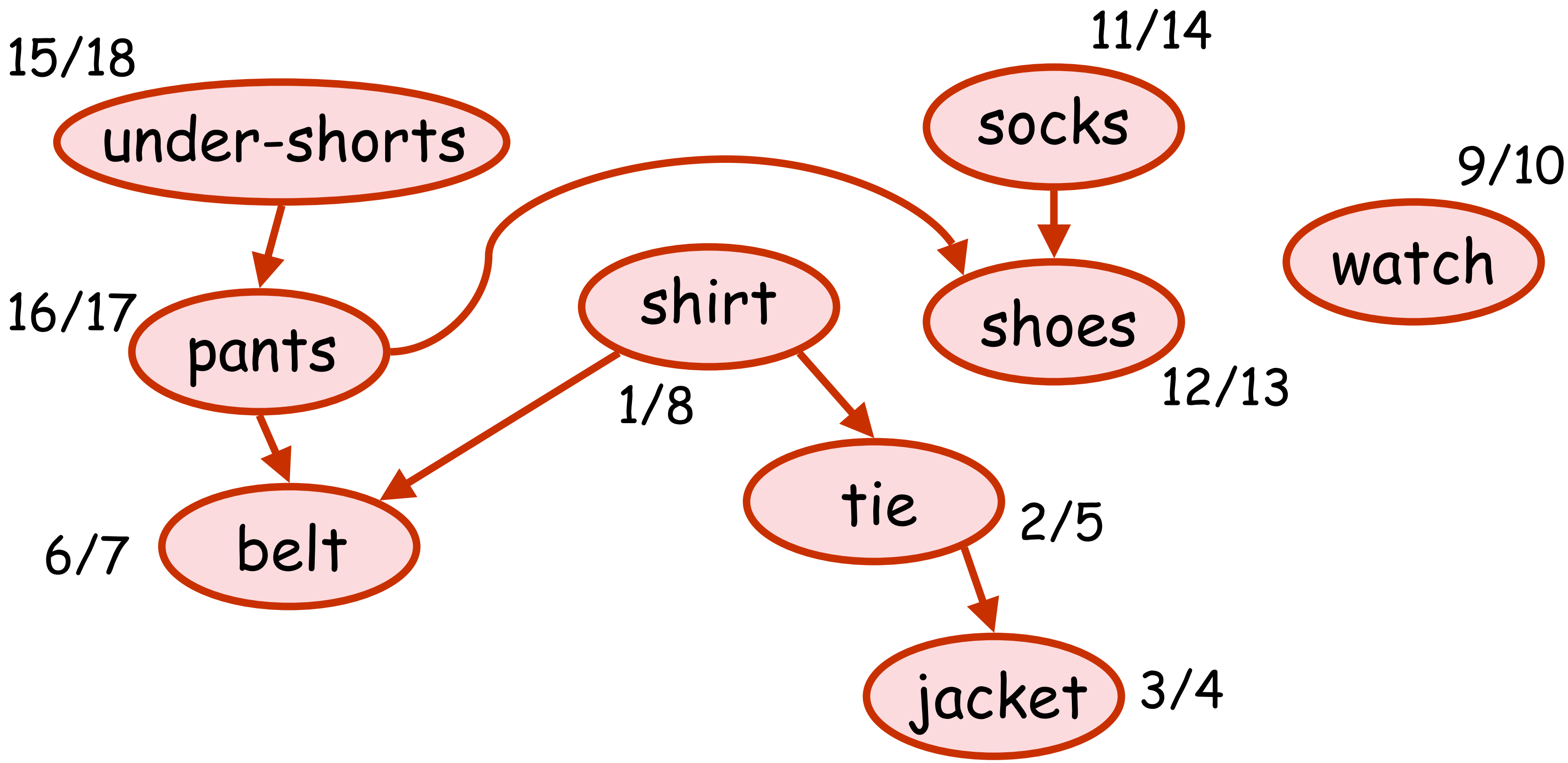
  ➔ $d(v) < d(u)$ (otherwise, by white-path, u must be an ancestor of v, so that (u,v) cannot be a cross edge)

  ➔ Since *G* is acyclic, v cannot reach u, so

  $$d(v) < f(v) < d(u) < f(u) \quad \text{(why??)}$$

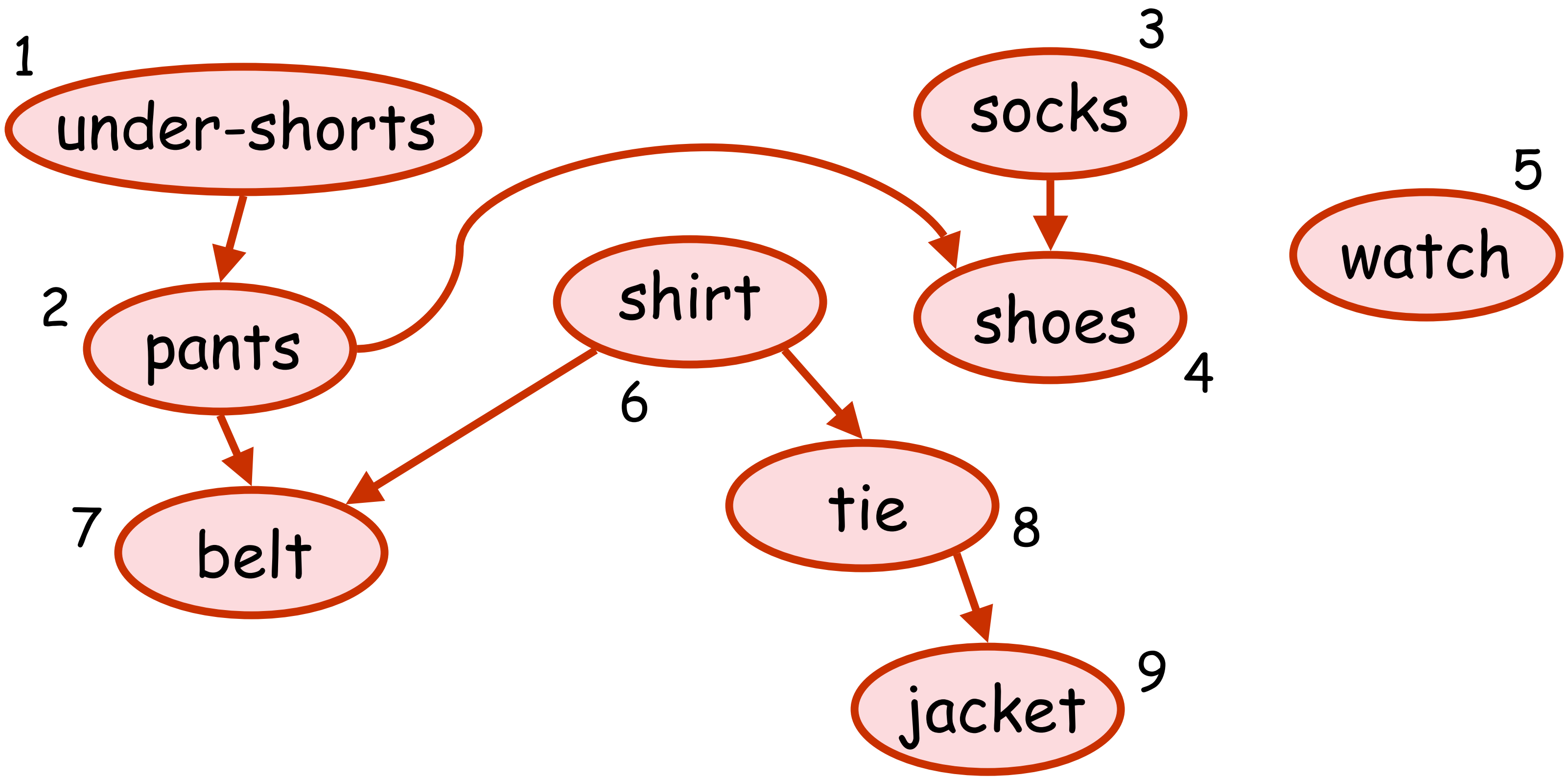
- Both cases show $f(u) > f(v)$ ➔ Done !

Topological Sort (Example)



Discovery and Finishing Times
after a possible DFS

Ordering Finishing Times (in descending order)



If we order the events from left to right, anything special about the edge directions ?

Performance

- Let $G = (V,E)$ be the input directed graph
- Running time for Topological-Sort :
  1. Perform DFS : $O(|V|+|E|)$ time
  2. Sort finishing times

     Naïve method: $O(|V| \log |V|)$ time

     Clever method: (use an extra stack S)

       During DFS, push a node into stack S once finished → no need to sort !!
- Total time: $O(|V|+|E|)$