# CS4311
# Design and Analysis of Algorithms

Lecture 23:

Elementary Graph Algorithms II

# About this lecture

- Depth First Search
  - DFS Tree and DFS Forest

- Properties of DFS
  - Parenthesis theorem (very important)
  - White-path theorem (very useful)

# Depth First Search (DFS)

- An alternative algorithm to find all vertices reachable from a particular source vertex $s$

- Idea:

  Explore a branch as far as possible before exploring another branch

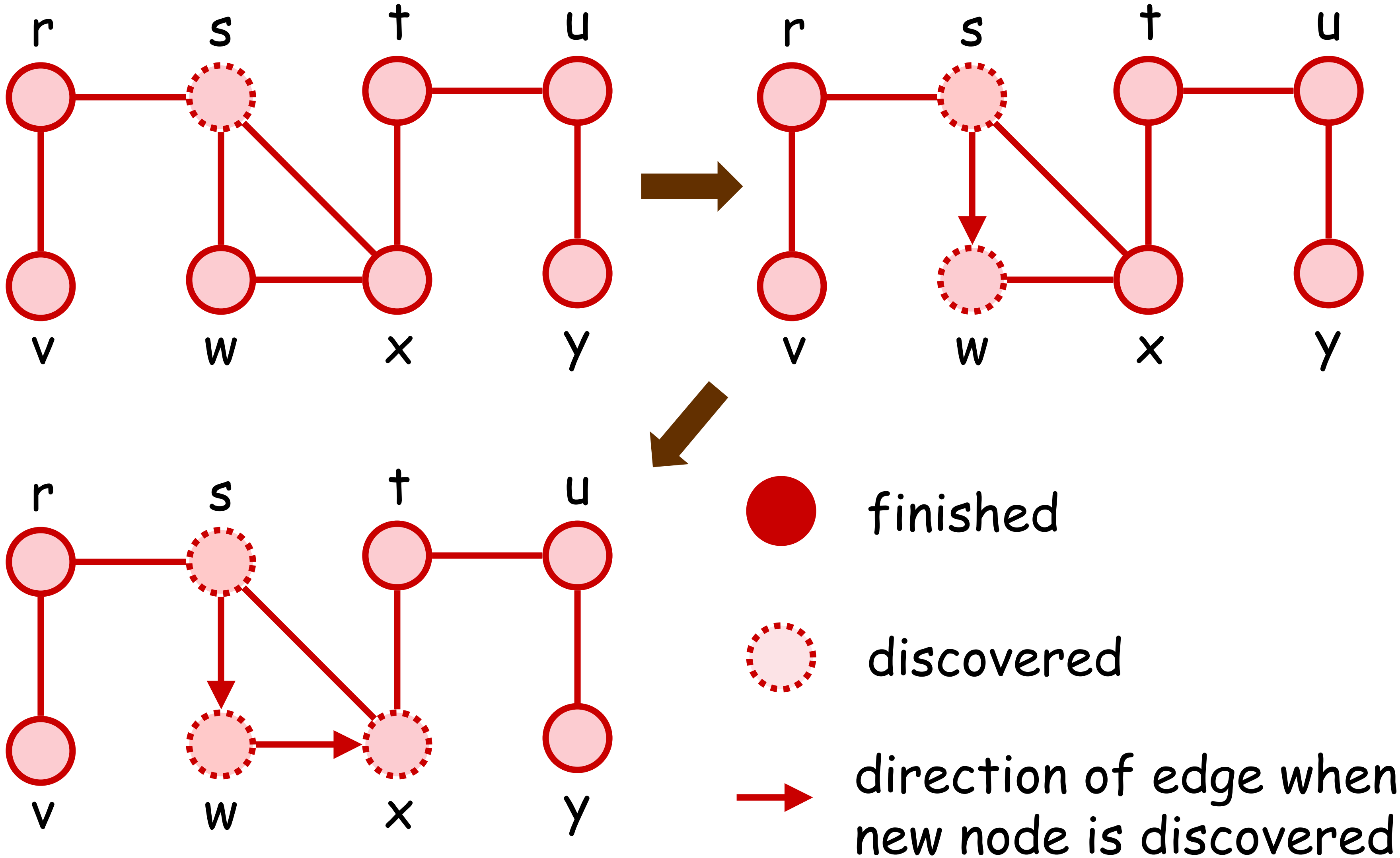
- Easily done by recursion or stack

# The DFS Algorithm

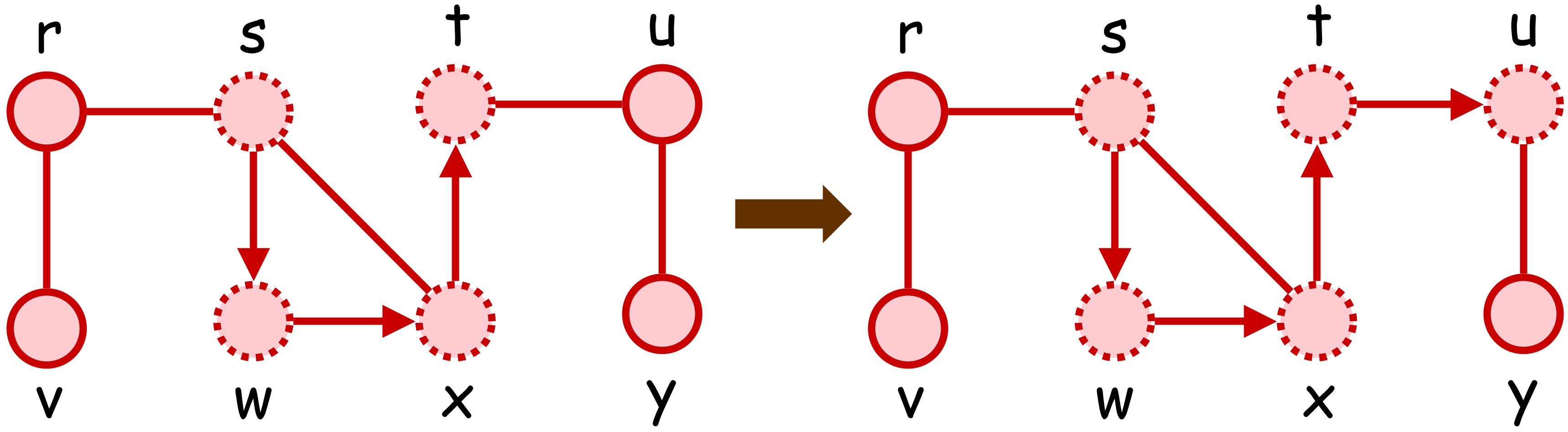
```
DFS(u)
{    Mark u as discovered ;
     while (u has unvisited neighbor v)
             DFS(v);
     Mark u as finished ;
}
```

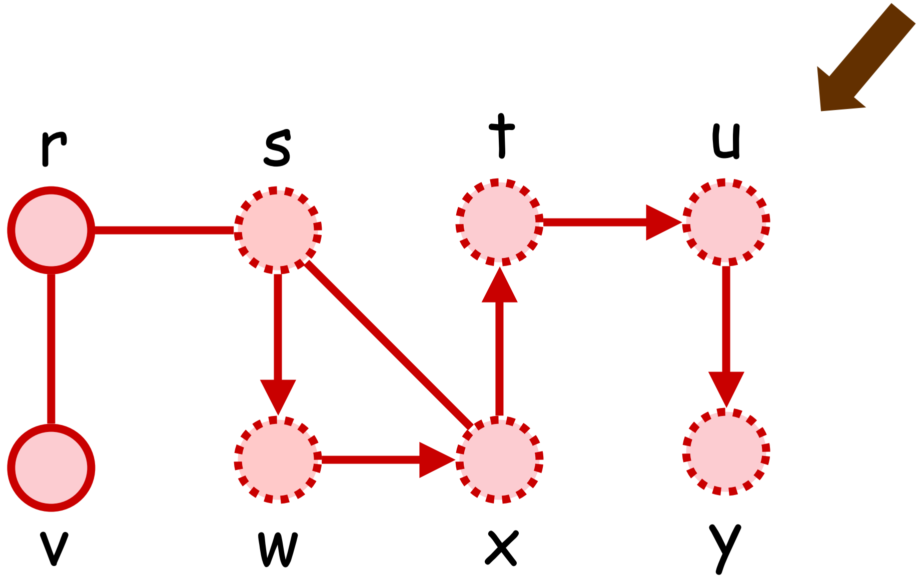
The while-loop explores a branch as far as possible before the next branch

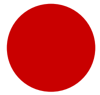
# Example (s = source)

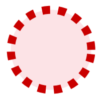


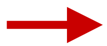
# Example (s = source)



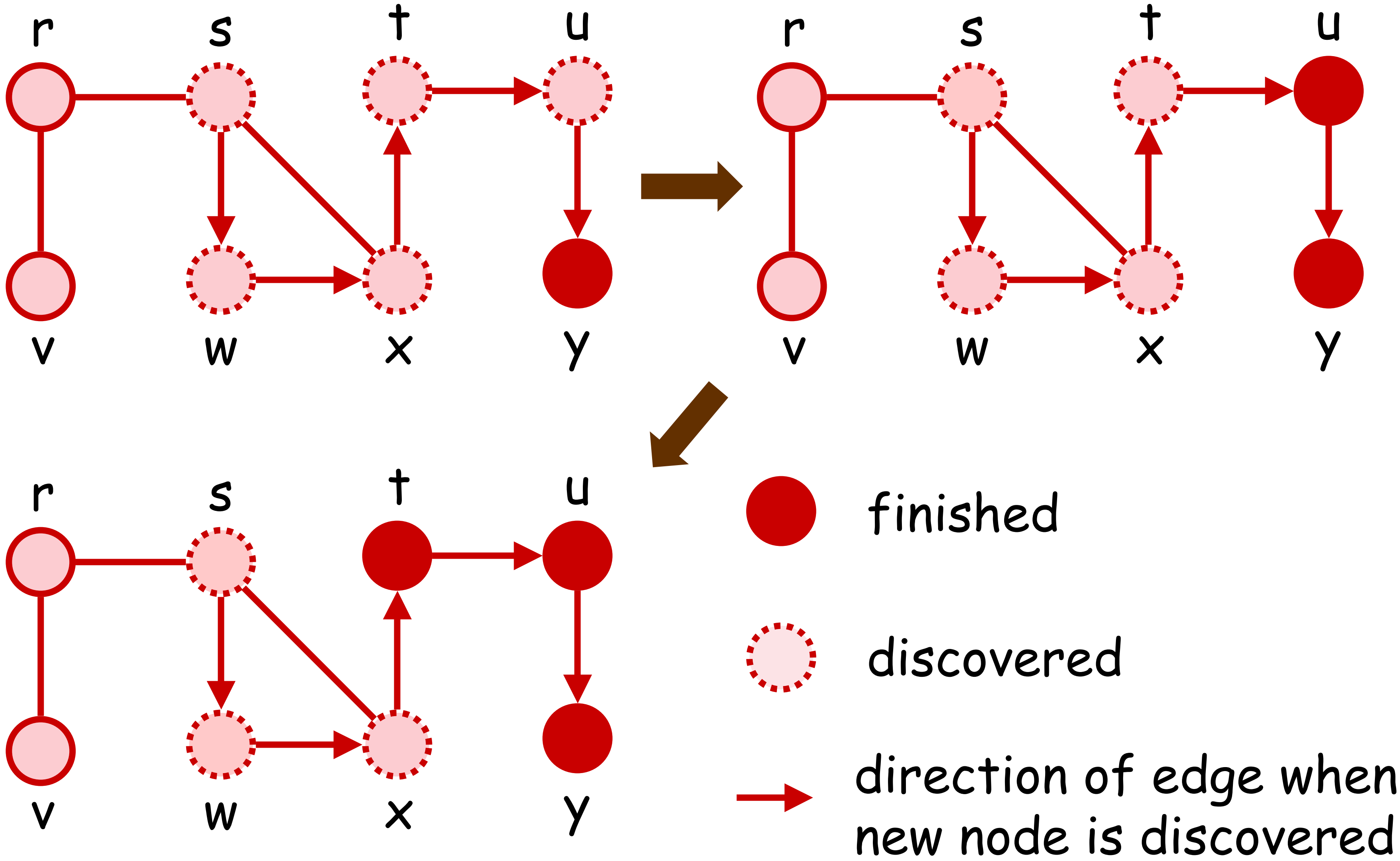




 finished

 discovered

 direction of edge when new node is discovered

# Example (s = source)



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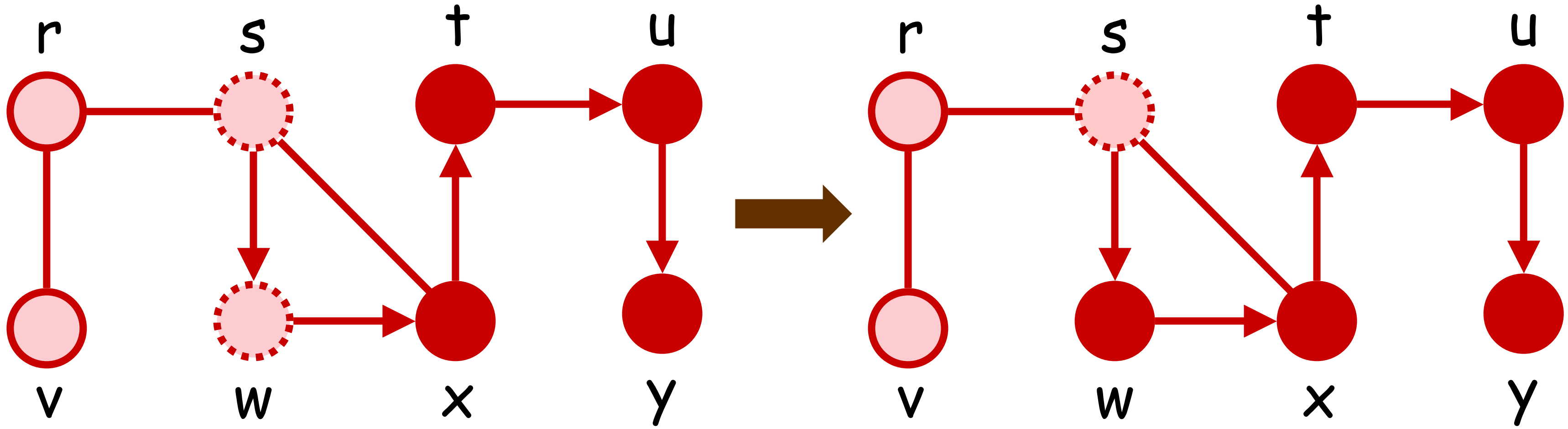
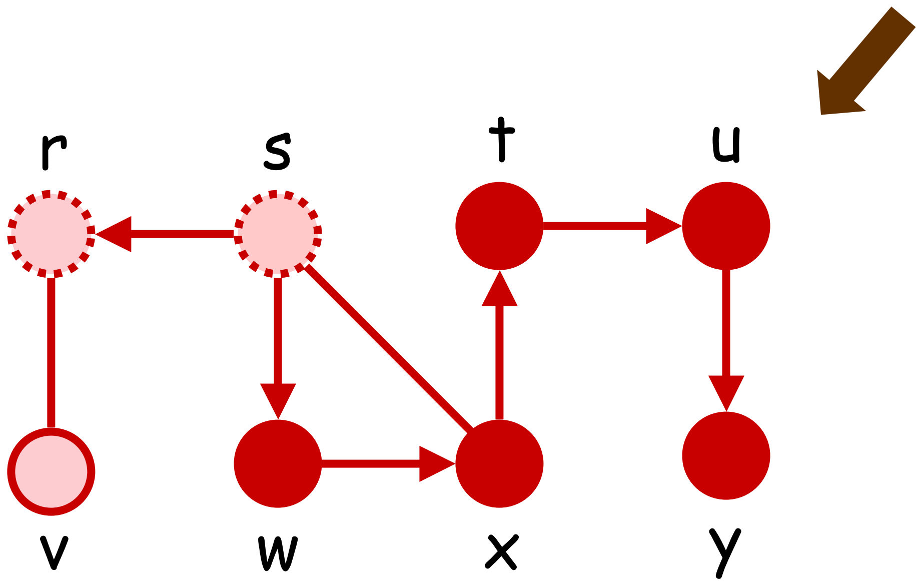
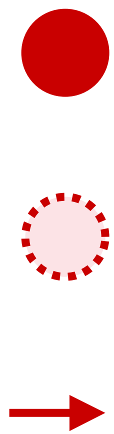


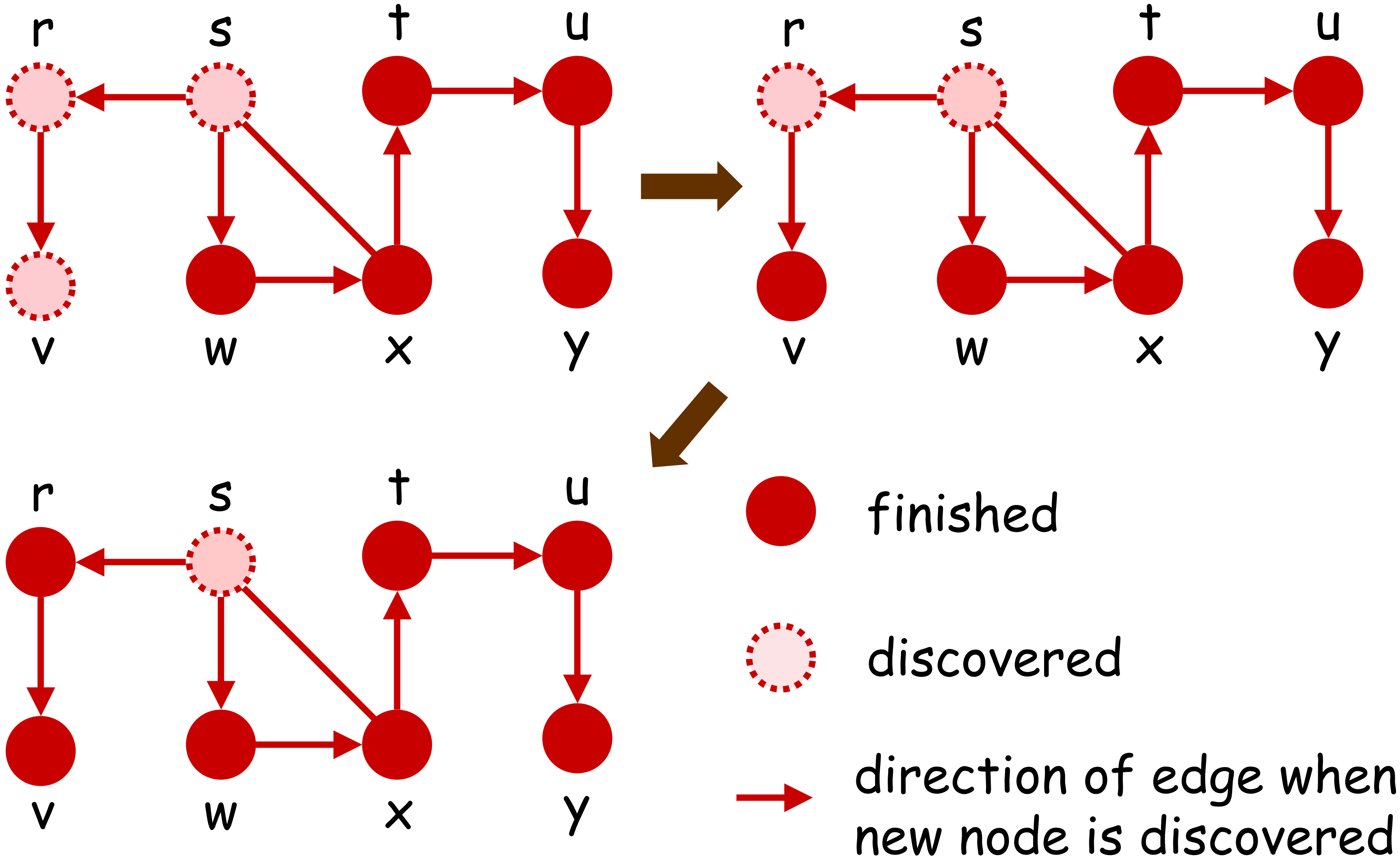



finished

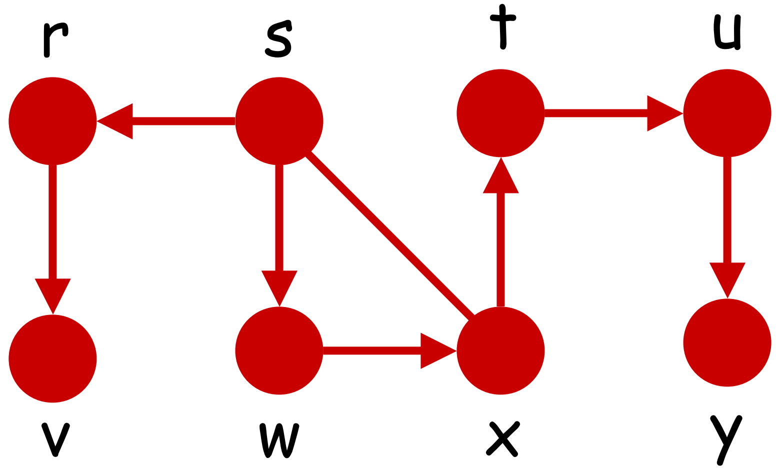
discovered

direction of edge when new node is discovered

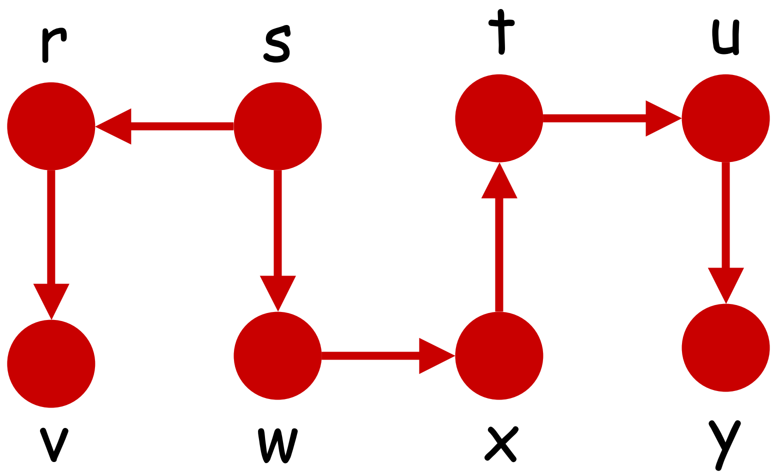
# Example (s = source)



# Example (s = source)




Done when s is discovered




The directed edges form a tree that contains all nodes reachable from s

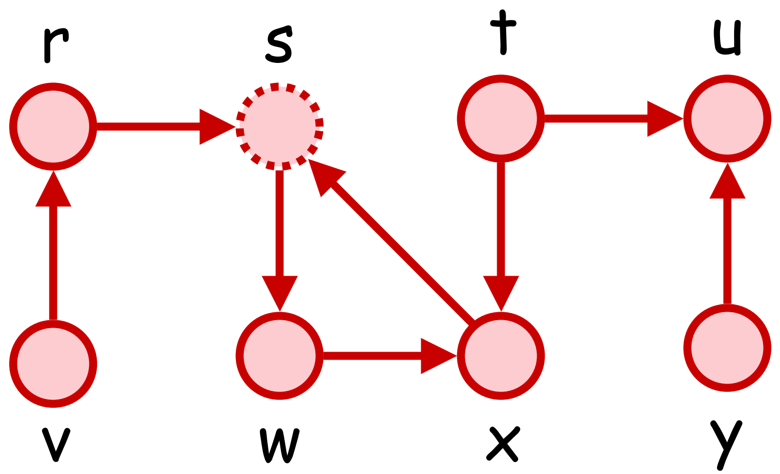
Called DFS tree of s

# Generalization

- Just like BFS, DFS may not visit all the vertices of the input graph G, because :
  - G may be disconnected
  - G may be directed, and there is no directed path from s to some vertex
- In most application of DFS (as a subroutine) , once DFS tree of s is obtained, we will continue to apply DFS algorithm on any unvisited vertices …

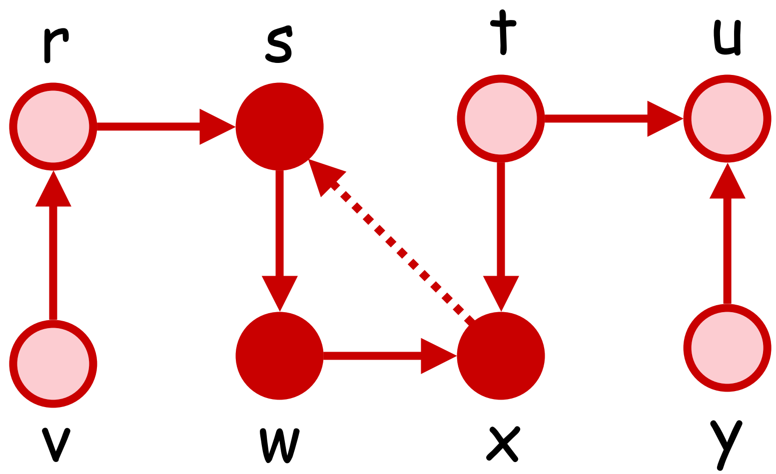
# Generalization (Example)

Suppose the input graph is directed



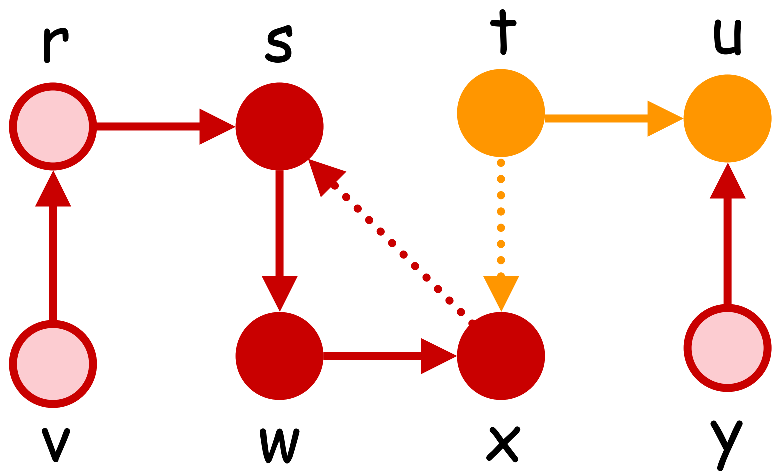
# Generalization (Example)

1. After applying DFS on s



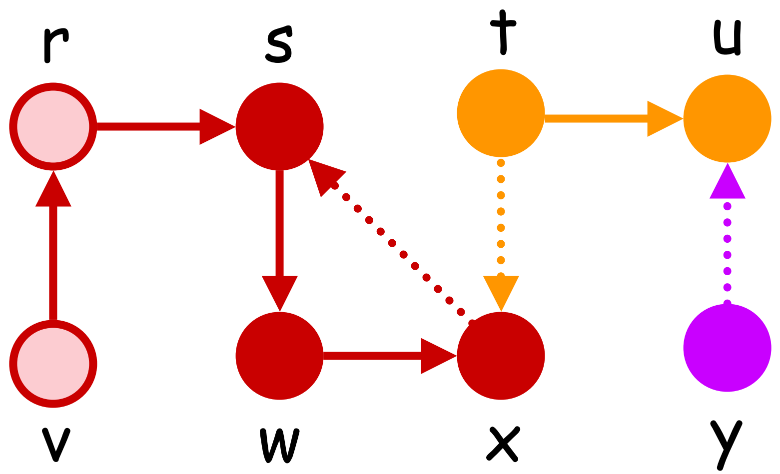
# Generalization (Example)

2. Then, after applying DFS on t



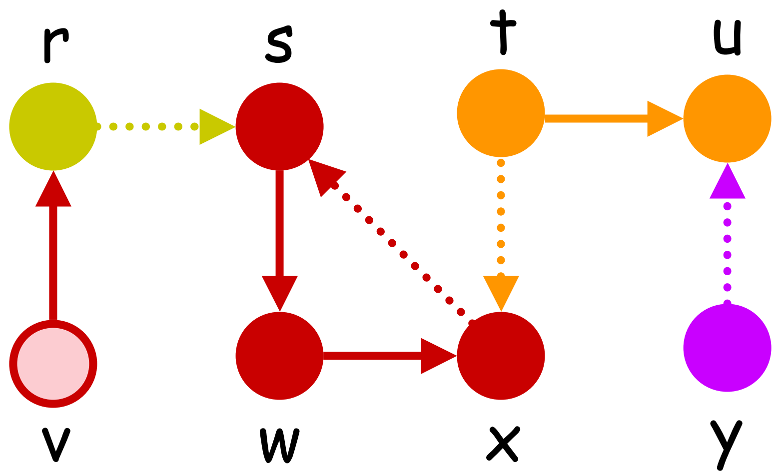
# Generalization (Example)

3. Then, after applying DFS on y



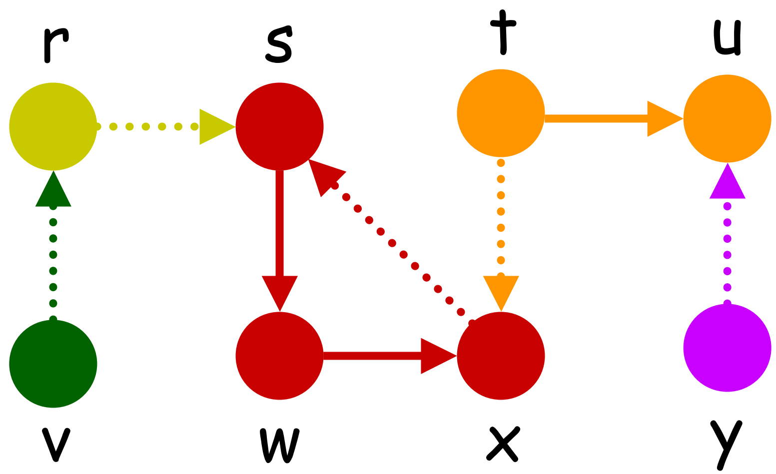
# Generalization (Example)

4. Then, after applying DFS on r



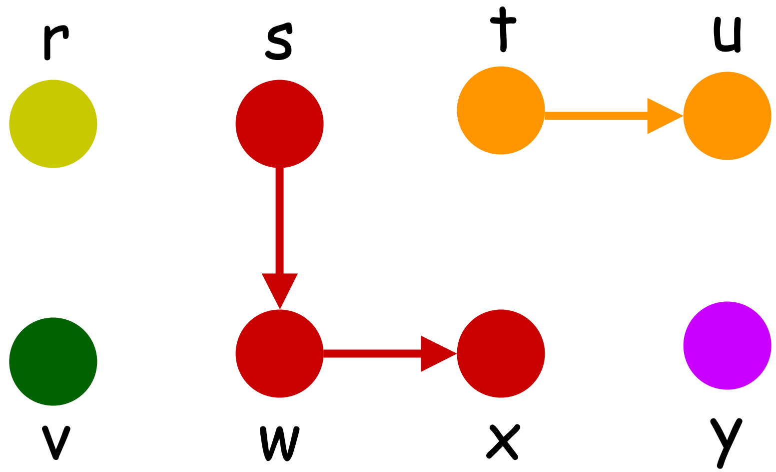
# Generalization (Example)

5. Then, after applying DFS on v



# Generalization (Example)

Result : a collection of rooted trees called DFS forest



# Performance

- Since no vertex is discovered twice, and each edge is visited at most twice (why?)
  ➔ Total time: $O(|V|+|E|)$

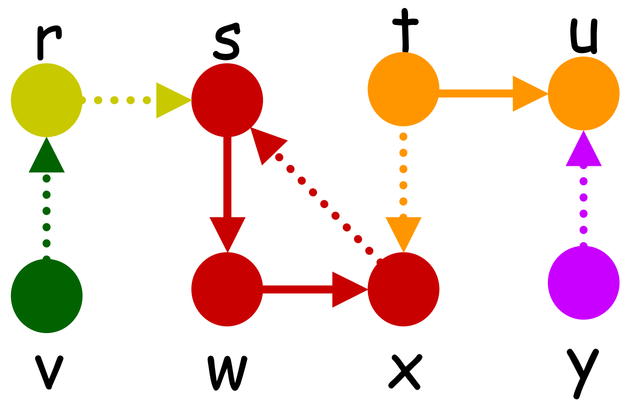
- As mentioned, apart from recursion, we can also perform DFS using a LIFO stack (Do you know how?)

# Who will be in the same tree ?

- Because we can only explore branches in an unvisited node
  - ➔ DFS(u) may not contain all nodes reachable by u in its DFS tree

E.g, in the previous run,

v can reach r, s, w, x but v 's tree does not contain any of them




Can we determine who will be in the same tree ?

# Who will be in the same tree ?

- Yes, we will soon show that by white-path theorem, we can determine who will be in the same tree as v at the time when DFS is performed on v

- Before that, we will define the discovery time and finishing time for each node, and show interesting properties of them

# Discovery and Finishing Times

- When the DFS algorithm is run, let us consider a global time such that the time increases one unit :
  - when a node is discovered, or
  - when a node is finished

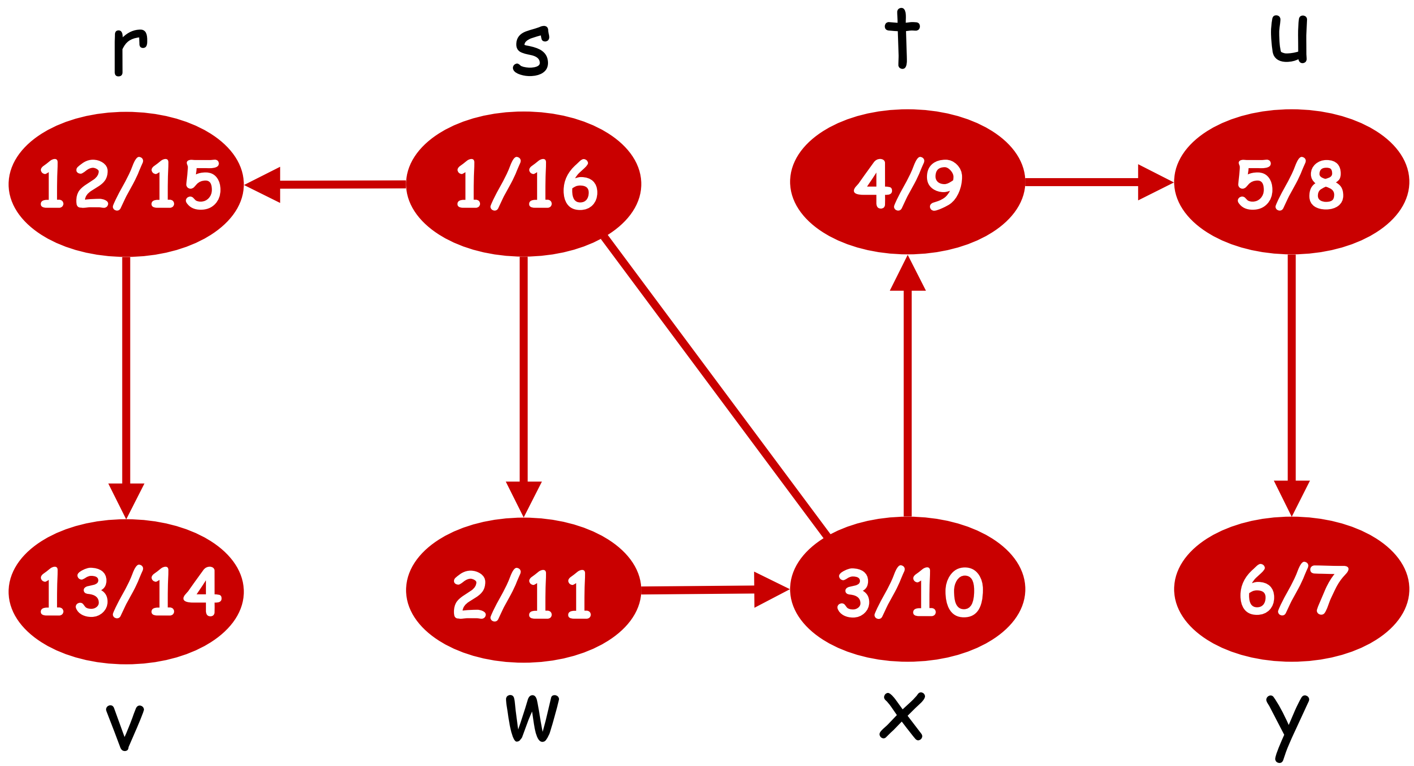
  (i.e., finished exploring all unvisited neighbors)

- Each node u records :

  d(u) = the time when u is discovered, and

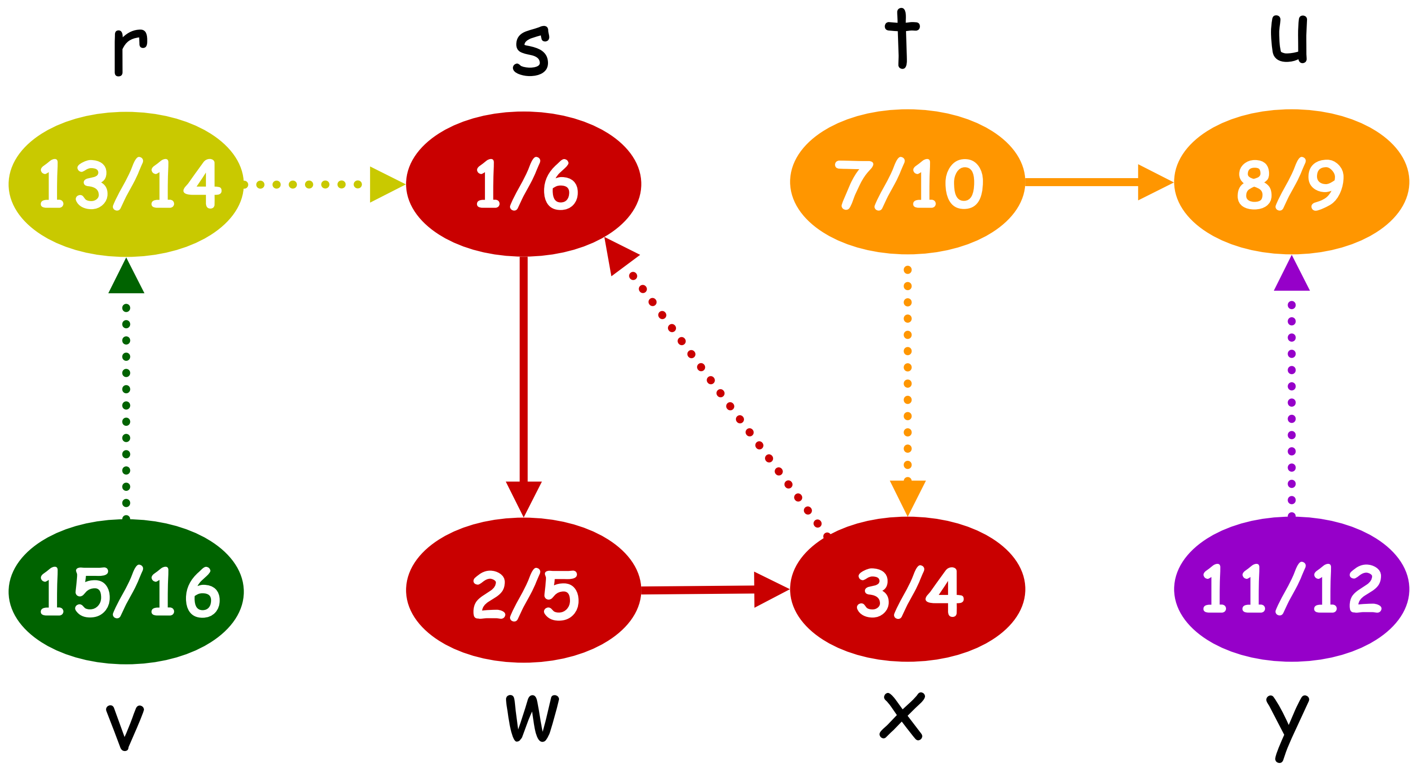
  f(u) = the time when u is finished

# Discovery and Finishing Times




In our first example
(undirected graph)

# Discovery and Finishing Times




In our second example
(directed graph)

# Nice Properties

Lemma: For any node $u$, $d(u) < f(u)$

Lemma: For nodes $u$ and $v$,

$d(u)$, $d(v)$, $f(u)$, $f(v)$ are all distinct

Theorem (Parenthesis Theorem):

Let $u$ and $v$ be two nodes with $d(u) < d(v)$ .

Then, either

1. $d(u) < d(v) < f(v) < f(u)$ [contain], or
2. $d(u) < f(u) < d(v) < f(v)$ [disjoint]

# Proof of Parenthesis Theorem

- Consider the time when v is discovered
- Since u is discovered before v, there are two cases concerning the status of u :
  - Case 1: (u is not finished)
    This implies v is a descendant of u
    ➔ $f(v) < f(u)$ (why?)
  - Case 2: (u is finished)
    ➔ $f(u) < d(v)$

# Corollary

**Corollary:**

v is a (proper) descendant of u

if and only if

$d(u) < d(v) < f(v) < f(u)$

Proof: v is a (proper) descendant of u

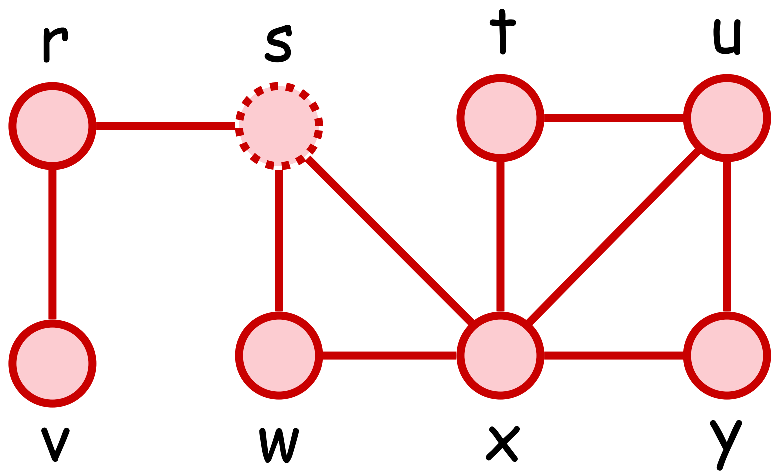
$\Leftrightarrow$ $d(u) < d(v)$ and $f(v) < f(u)$

$\Leftrightarrow$ $d(u) < d(v) < f(v) < f(u)$

# White-Path Theorem

Theorem: By the time when DFS is performed on u, for any way DFS is done, the descendants of u are the same, and they are exactly those nodes reachable by u with unvisited (white) nodes only

E.g.,




If we perform DFS(w) now, will the descendant of w always be the same set of nodes?

# Proof (Part 1)

- Suppose that v is a descendant of u

  Let P = ($u$, $w_1$, $w_2$, …, $w_k$, $v$) be the directed path from u to v in DFS tree of u

  Then, apart from u, each node on P must be discovered after u

  → They are all unvisited by the time we perform DFS on u

  → Thus, at this time, there exists a path from u to v with unvisited nodes only

# Proof (Part 2)

- So, every descendant of u is reachable from u with unvisited nodes only

- To complete the proof, it remains to show the converse :

| |
|---|
| Any node reachable from u with unvisited nodes only becomes u 's descendant |

is also true
(We shall prove this by contradiction)

# Proof (Part 2)

- Suppose on contrary the converse is false
- Then, there exists some v, reachable from u with unvisited nodes only, does not become u 's descendant
  - If more than one choice of v, let v be one such vertex closest to u

➔ $d(u) < f(u) < d(v) < f(v)$ … EQ.1

# Proof (Part 2)

- Let $P = (u, w_1, w_2, \ldots, w_k, v)$ be any path from $u$ to $v$ using unvisited nodes only
- By our choice of $v$ (closest one), all $w_1, w_2, \ldots, w_k$ become $u$'s descendants
- This implies:

$$d(u) \leq d(w_k) < f(w_k) \leq f(u)$$

Handle special case: when $u = w_k$

- Combining with EQ.1, we have

$$d(w_k) < f(w_k) < d(v) < f(v)$$

# Proof (Part 2)

- However, since there is an edge (no matter undirected or directed) from $w_k$ to $v$,
  if $d(w_k) < d(v)$ , then we must have
  $$d(v) < f(w_k) \qquad \ldots \text{ (why??)}$$
- Consequently, it contradicts with :
  $$d(w_k) < \underline{f(w_k) < d(v)} < f(v)$$

➔ Proof completes