CS4311
Design and Analysis of Algorithms

Lecture 21:

Data Structures for Disjoint Sets II

About this lecture

- Data Structure for Disjoint Sets
  - Support Union and Find operations

- Various Methods:
  1. Linked List
  2. Union by Size
  3. Union by Rank
  4. Union by Rank + Path Compression

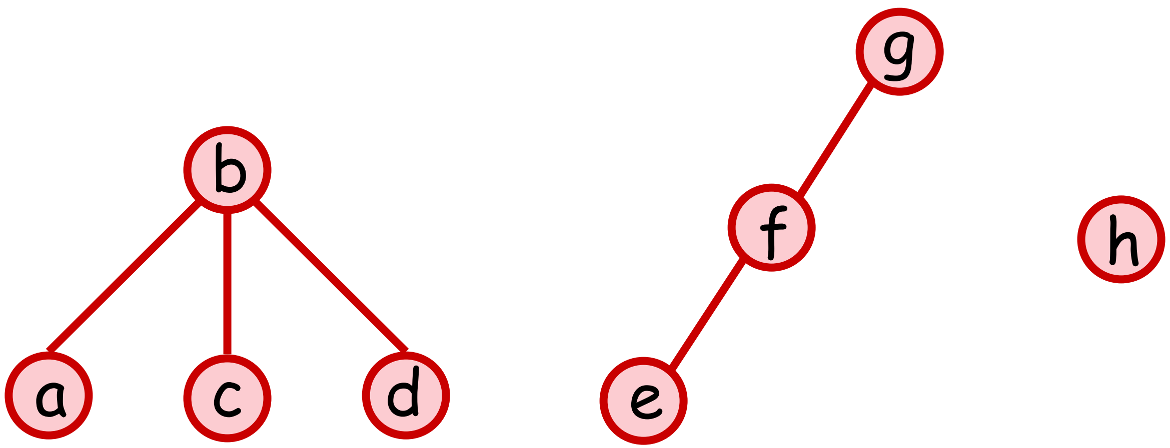
(this lecture)

Disjoint-Set Forest

- Another popular method to maintain disjoint sets is by a forest
  - Each set ⇔ a separate rooted tree
  - Representative ⇔ root of tree
- Unlike the linked lists implementation, each element now points only to its parent (and does not directly point to the representative)

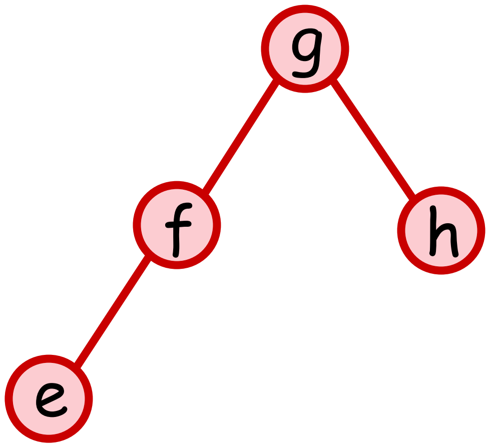
Example

Current dynamic sets : { {a,b,c,d}, {e,f,g}, {h} }



Disjoint-Set Forest

- To perform Union(x,y), we join the trees containing x and containing y, by linking their roots
- E.g. Union(f,h) in previous example gives:



Disjoint-Set Forest

- Let H_{max} = max height of all trees
- In the worst-case:

Make-Set :	$\Theta(1)$ time
Find or Union :	$O(H_{max})$ time

➔ m operations on n elements :

worst-case $\Theta(mn)$ time

Union By Size

- Let us apply a union-by-size heuristic :

To perform Union, we link root of the smaller tree to root of the larger tree

➔ $H_{max} = O(\log n)$ (how to prove??)

➔ m operations : $\Theta(m \log n)$ time

Union By Rank

- A similar heuristic is called union-by-rank
- Each node keeps track of its rank – an upper bound on the height of the node
  - In a single-node tree (created by Make-Set)

rank of root = 0

To perform Union, we link root with smaller rank to root with larger rank

Union By Rank

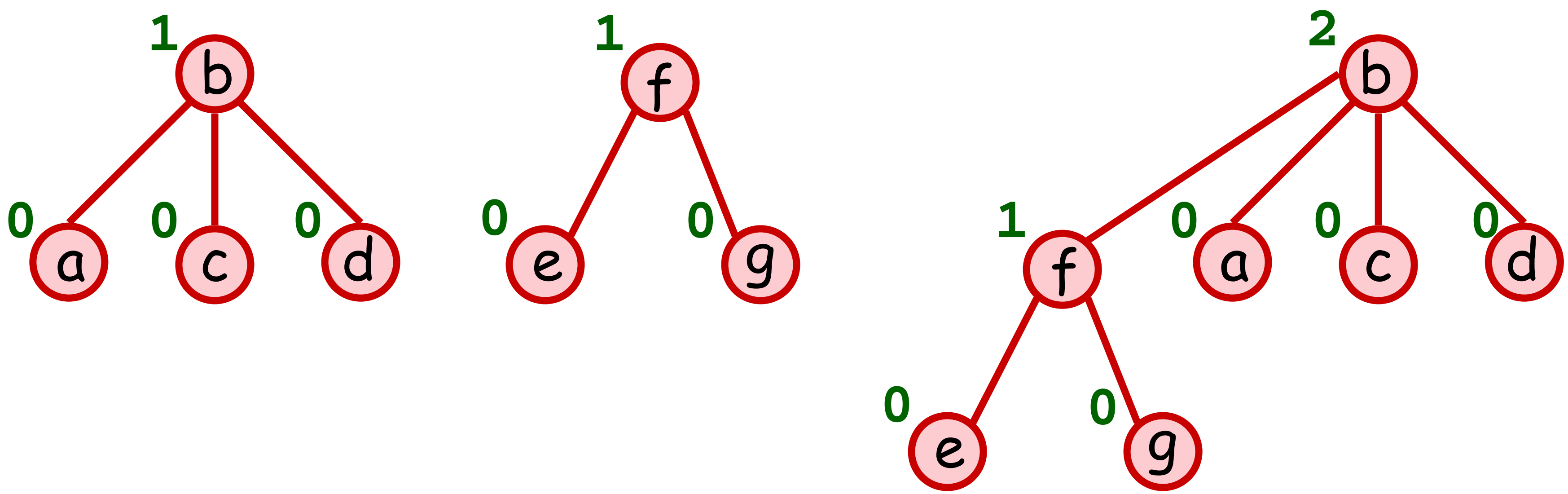
- Rank needs not be very accurate
  - as long as it always gives an upper bound of height is enough
- When Union is performed, only the rank of the roots may change :
  - If both roots have same rank
    ➔ rank of new root increases by 1
  - Else, no change

Example

? = rank

Before Union

After Union(c,f)



Union By Rank

- Let H_{max} = max height of all trees

  ➔ $H_{max} = O(\log n)$ (how to prove??)

  ➔ m operations : $\Theta(m \log n)$ time

- So, union by rank is no better than union by size, but …

Path Compression

- The closer a node to its root, the faster the Find or Union operation
- When we perform Find(x), we will need to find the root of the tree containing x

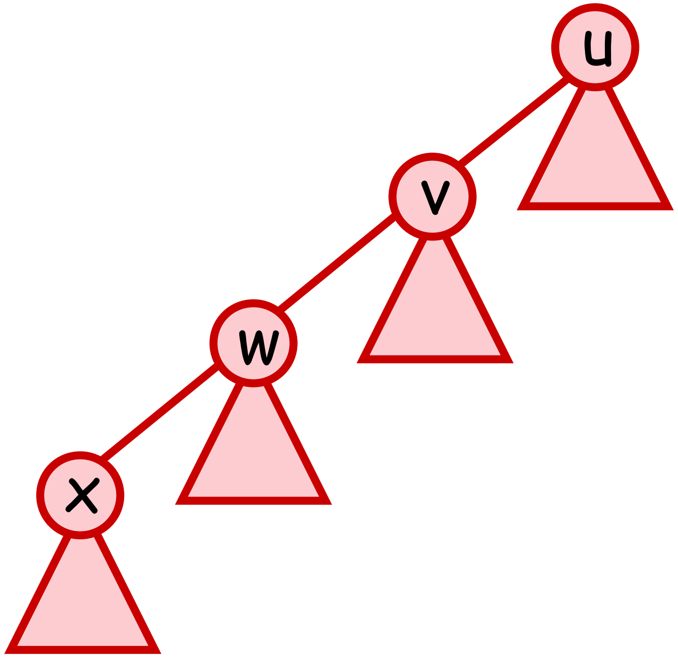
  ➔ will access every ancestor of x

  - why don't we make all these ancestors of x closer to the root now?

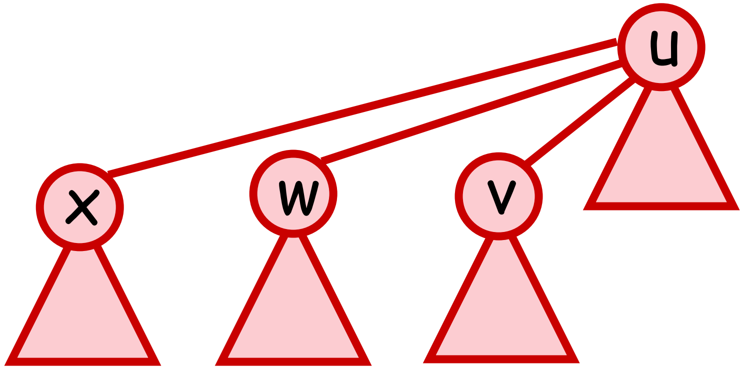
    (Because no increase in asymptotic performance !!!)

Example

Before Find(x)



After Find(x)



Union by Rank + Path Compression

- With path compression, the more Find we have, the better its average performance

Fact: In the worst-case, m operations with f Find runs in $\Theta(m + f \log_{2+f/n} n)$ time

➔ still no improvement in worst-case

- Interestingly, by combining union-by-rank (at Union) and path compression (at Find)

m operations: $\Theta(m\,\alpha(n))$ time

Inverse Ackermann
(in practice, at most 4)

Finding Connected Components

- Recall: To find connected components of a graph G with n vertices and m edges
  - there are n Make-Set and m Find or Union operations

- Which scheme for dynamic disjoint sets gives the best running time (theoretically) ?

Ans. Depends on m (why?)