CS4311 Design and Analysis of Algorithms

Lecture 20:

Data Structures for Disjoint Sets I

About this lecture

- Data Structure for Disjoint Sets
  - Support Union and Find operations

- Various Methods:

1. Linked List (this lecture)
2. Union by Size
3. Union by Rank
4. Union by Rank + Path Compression

Maintaining Disjoint Set

- In some applications, especially in algorithms relating to graphs, we often have a set of elements, and want to maintain a dynamic partition of them
    - I.e., the partition changes over time

- Our target corresponds to maintaining dynamic disjoint sets of the elements

Maintaining Disjoint Set

- Let $\Sigma = \{ S_1, S_2, \ldots, S_k \}$ be a collection of dynamic disjoint sets of the elements
- Let x and y be any two elements
- We want to support:

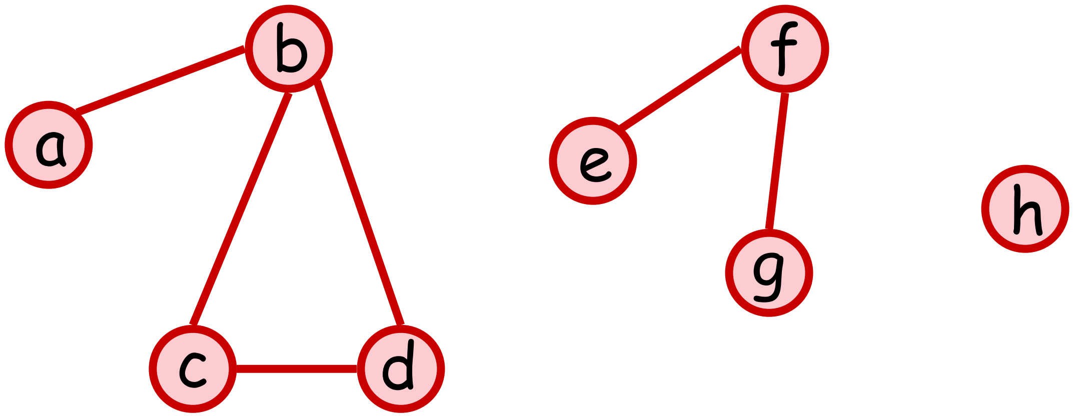
Make-Set(x): create a set containing x

Find(x) : return which set x belongs

Union(x,y) : merge the sets containing x and containing y into one

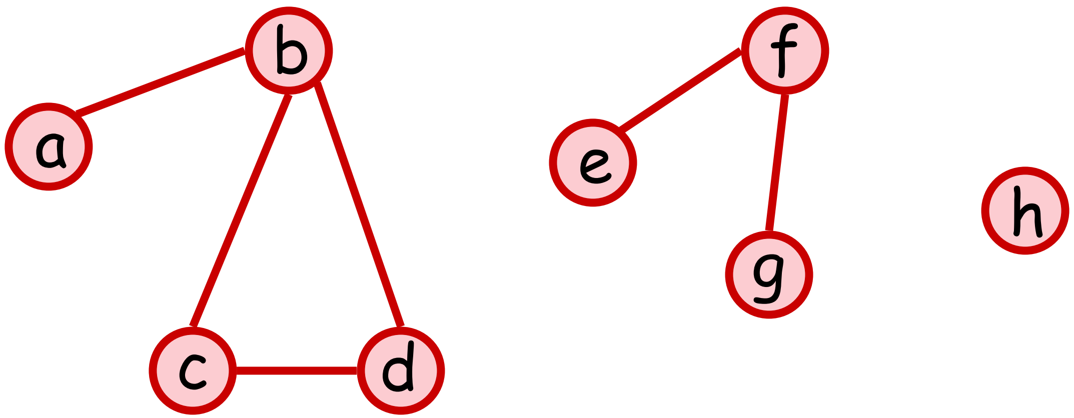
Example Application: Finding Connected Components

Step 0: Begin with the input graph



Example Application: Finding Connected Components

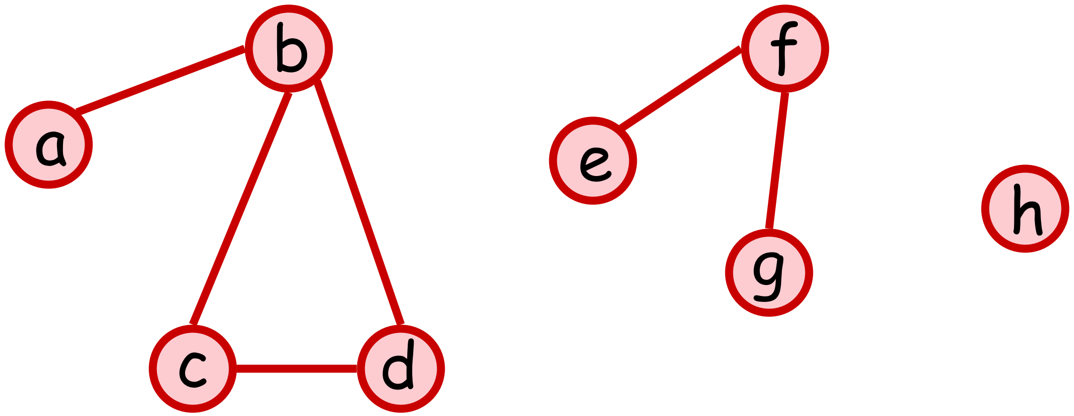
Step 1: Make-Set(v) for each vertex v



current Σ: { {a}, {b}, {c}, {d}, {e}, {f}, {g}, {h} }

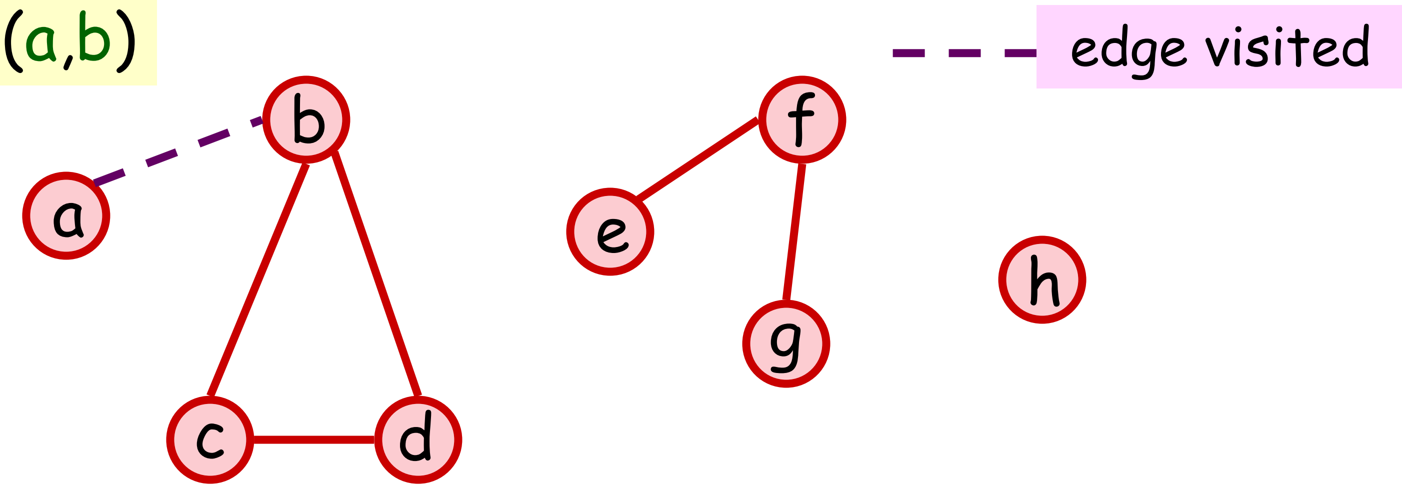
Example Application: Finding Connected Components

Step 2: Visit each edge (u,v), perform Union(u,v)



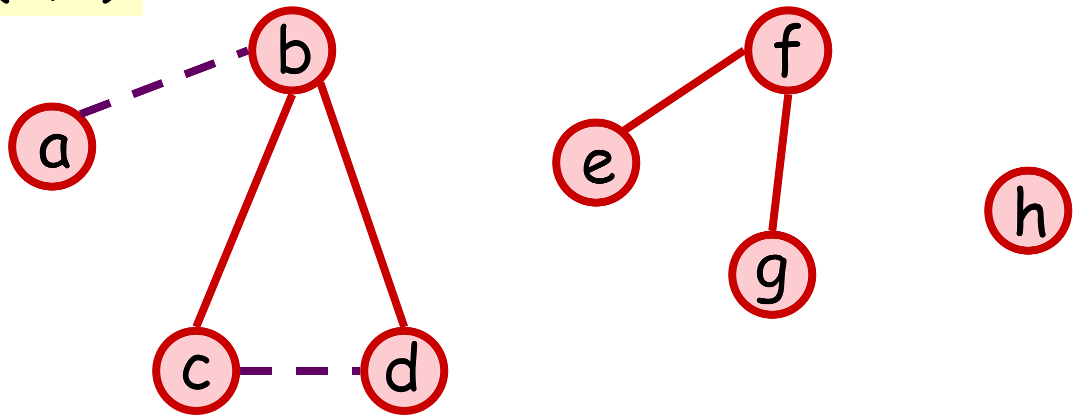
current Σ: $\{\ \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}\ \}$

Step 2: Visit (a,b)



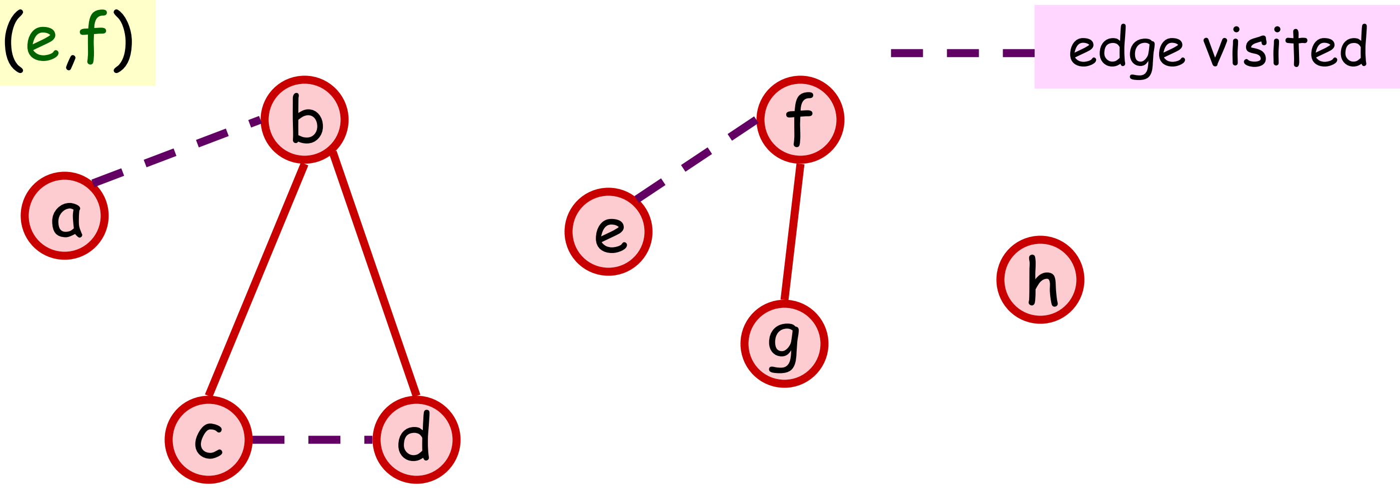
current Σ: { {a,b}, {c}, {d}, {e}, {f}, {g}, {h} }

Step 2: Visit (c,d)



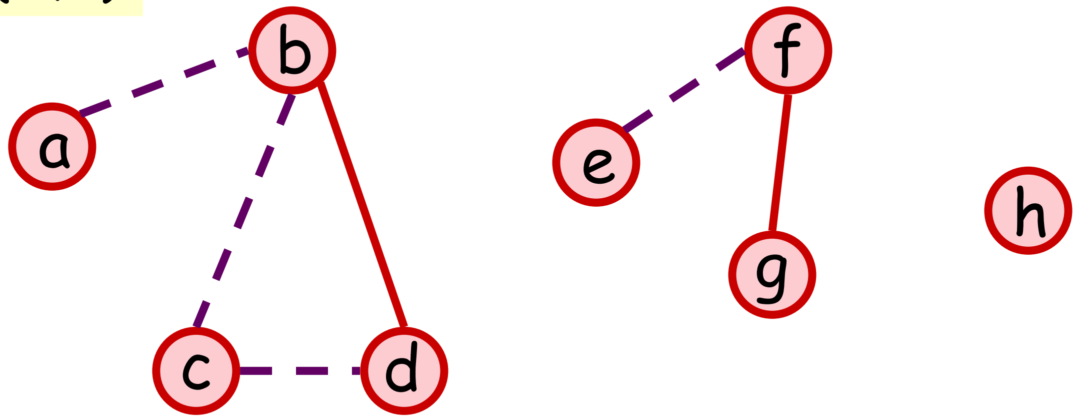
current Σ: { {a,b}, {c,d}, {e}, {f}, {g}, {h} }

Step 2: Visit (e,f)



current Σ: { {a,b}, {c,d}, {e,f}, {g}, {h} }

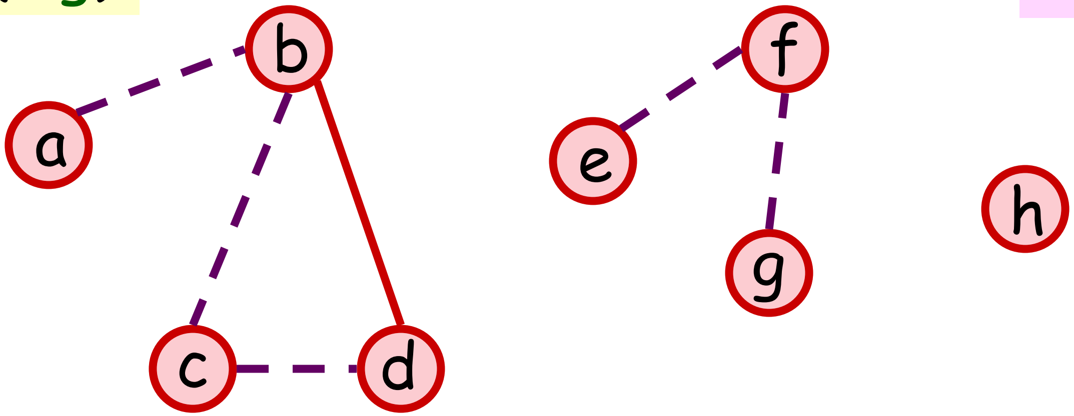
Step 2: Visit (b,c)



current Σ: { {a,b,c,d}, {e,f}, {g}, {h} }

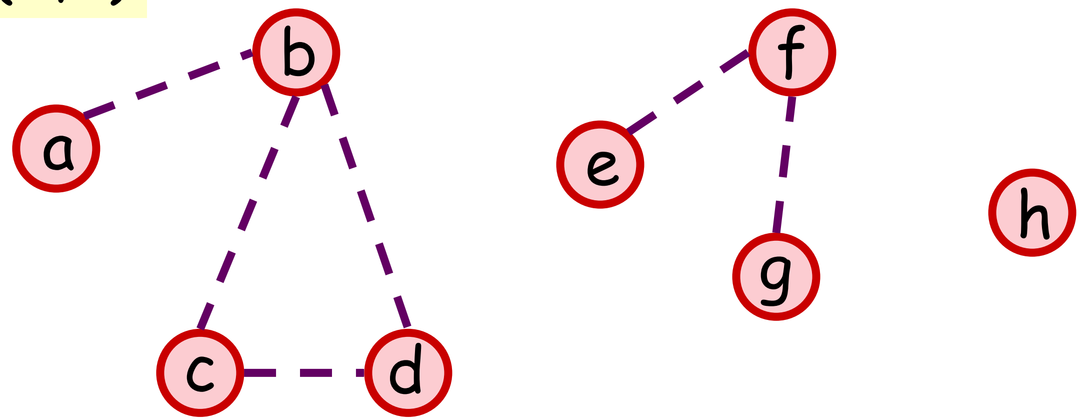
Step 2: Visit (f,g)

edge visited



current Σ: { {a,b,c,d}, {e,f,g}, {h} }

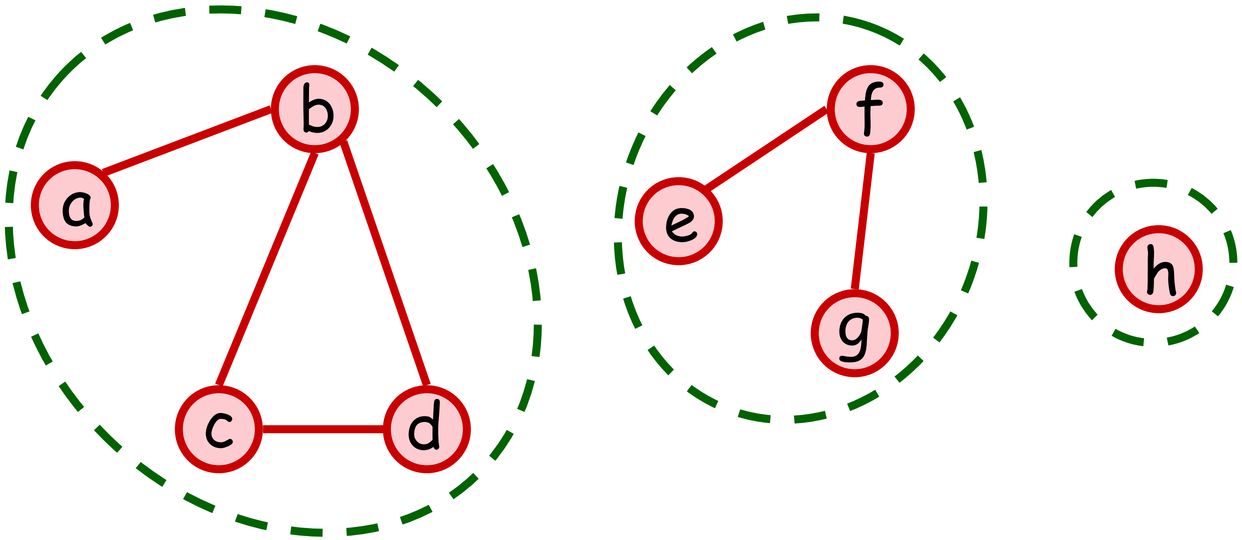
Step 2: Visit (b,d)



current Σ: { {a,b,c,d}, {e,f,g}, {h} }

Example Application: Finding Connected Components

After Step 2 (when all edges visited) :
Each Disjoint Set ⇔ Connected Component



current Σ: { {a,b,c,d}, {e,f,g}, {h} }

Remarks

- To facilitate Find(x), each set usually chooses one of its element as a representative

  ➔ Find(x) returns the representative element of the set where x belongs

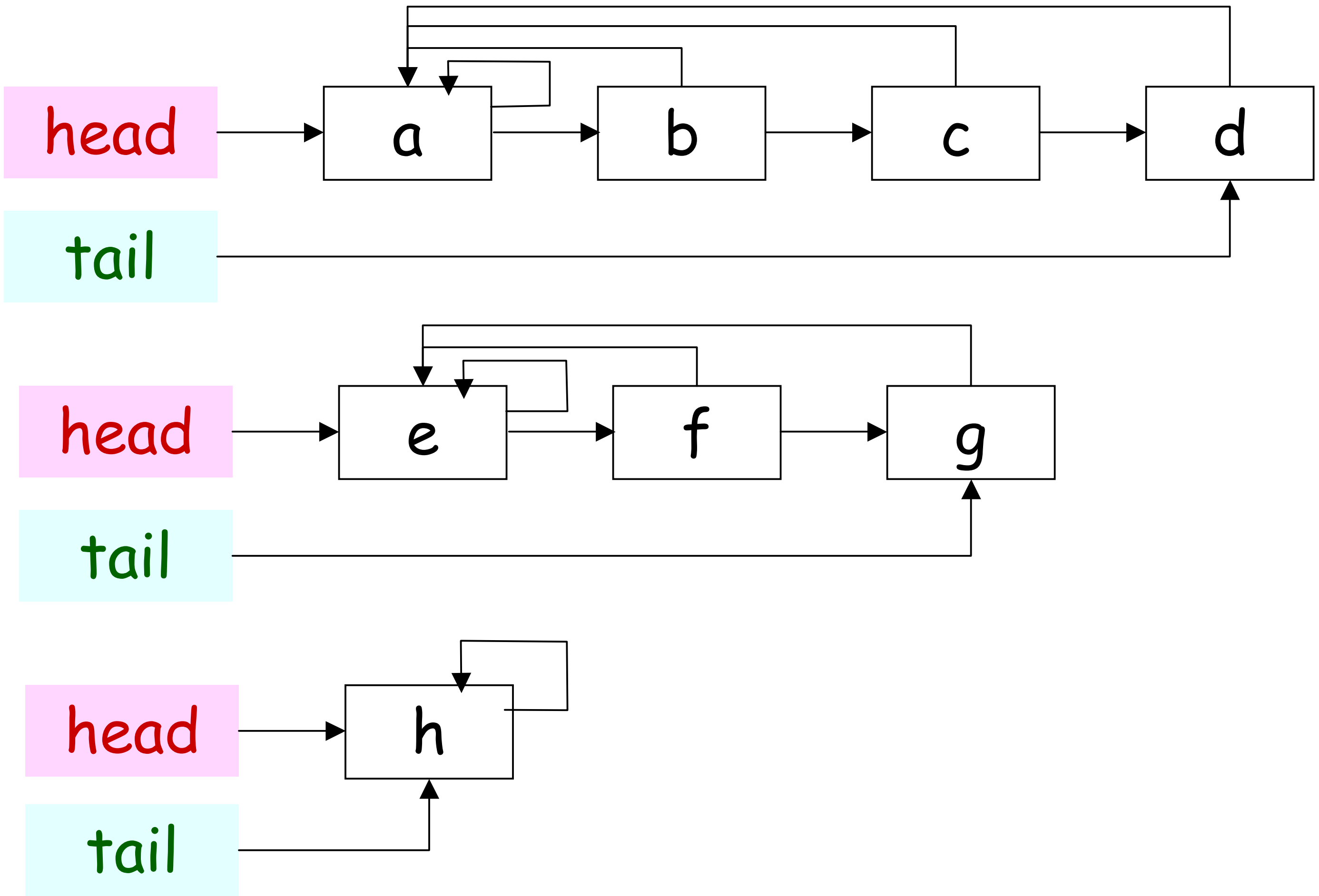
- To check if x and y belong to the same set, we can just check if

  Find(x) == Find(y)

Disjoint Set with Linked List

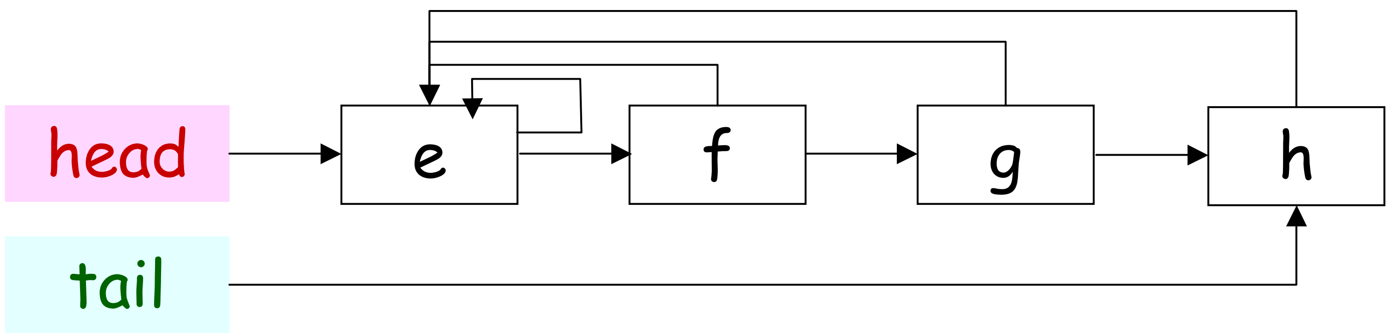
- A simple way to maintain disjoint sets is by using linked lists:
  - Each set ⇔ a separate linked list
  - Representative ⇔ head element of list
- To facilitate Find and Union:
  - each element in the list has an extra pointer that points at head element
  - each list has a pointer to the tail

E.g., disjoint sets { {a,b,c,d}, {e,f,g}, {h} } is stored by:



Disjoint Set with Linked List

- To perform Union(x,y), we join the lists containing x and containing y, one list after the other, and update the pointers of the latter list
- E.g. Union(g,h) in previous example gives:



Disjoint Set with Linked List

- In the worst-case:

  Make-Set or Find : $\Theta(1)$ time

  Union : $\Theta(n)$ time

➔ m operations on n elements :

$\Theta(m + n^2)$ time

Disjoint Set with Linked List

- Let us apply a weighted-union heuristic :

  To perform Union, we merge lists with longer one first, followed by shorter list

- No change in worst-case time, but …
- m operations : $\Theta(m + n \log n)$ time

Reason: The time to perform Union is from changing head pointer of each element in the latter list

With the heuristic, each element changes head pointer at most $\log n$ times (why??)