CS4311
Design and Analysis of Algorithms

Lecture 19: Fibonacci Heap II

About this lecture

- Decrease-Key & Delete in Fibonacci Heap
  → Based on cutting some node from its parent, and a simple rule which decides if further cuts are needed

- Bounding MaxDeg(n)

Rule of Further Cuts

- Let x be a node with a parent node y, such that at some time, x was a root and was then linked to y

The rule is as follows:

> After the above linking event,
> as soon as x has lost its second child,
> we cut x from y, making it a new root

Rule of Further Cuts

- To help us keep track of the status of each node, we use marking in a node !!!
  - we mark a non-root node x if it has lost the first child

    ➔ If a non-root marked node loses a child, it is cut from its parent
  - we unmark a node x if
    (i) it becomes a new root [after a cut], or
    (ii) it receives a parent [after Extract-Min]

Decrease-Key

- Decrease-Key(H, x, k):

  Report error if k $>$ key of x;

  Update the key of x to k ;

  ```
  /* fix if min-heap property is violated */
  ```

  if (x $\neq$ root and x's key $<$ its parent's key)

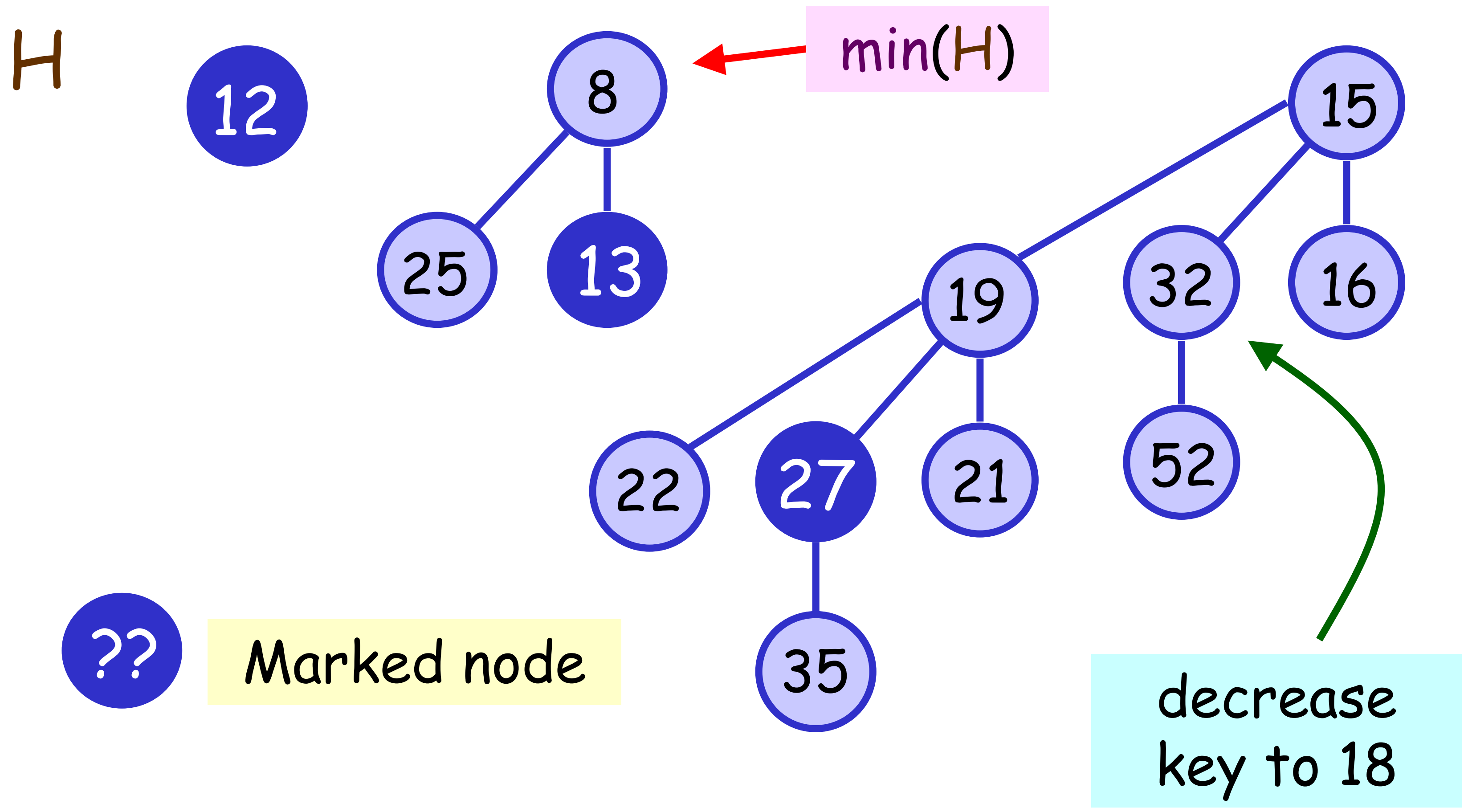
  { Cut x from its parent;

  Perform further cuts (recursively); }

  Update min[H] if needed

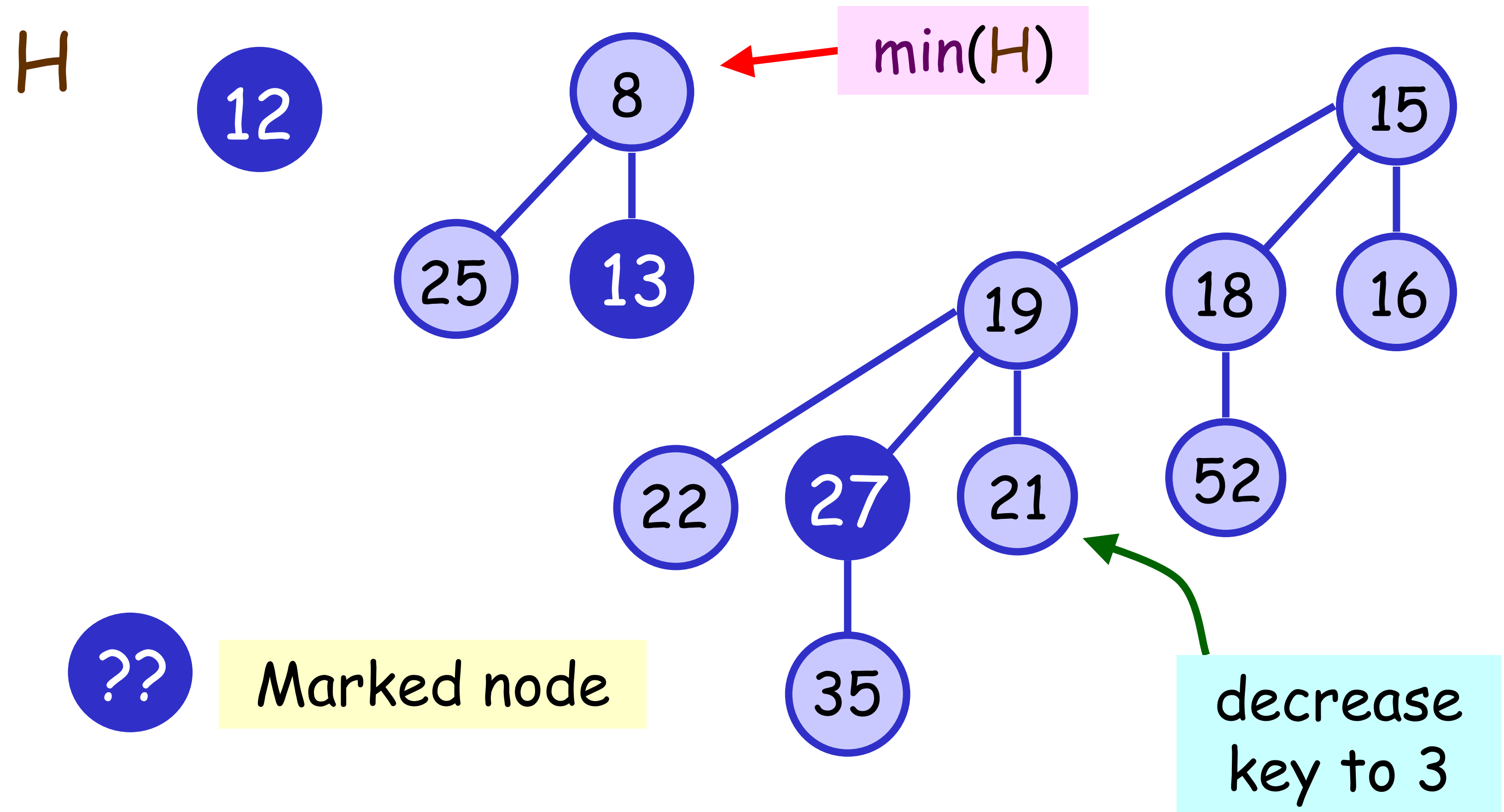
Decrease-Key (Example)

Before Decrease-Key



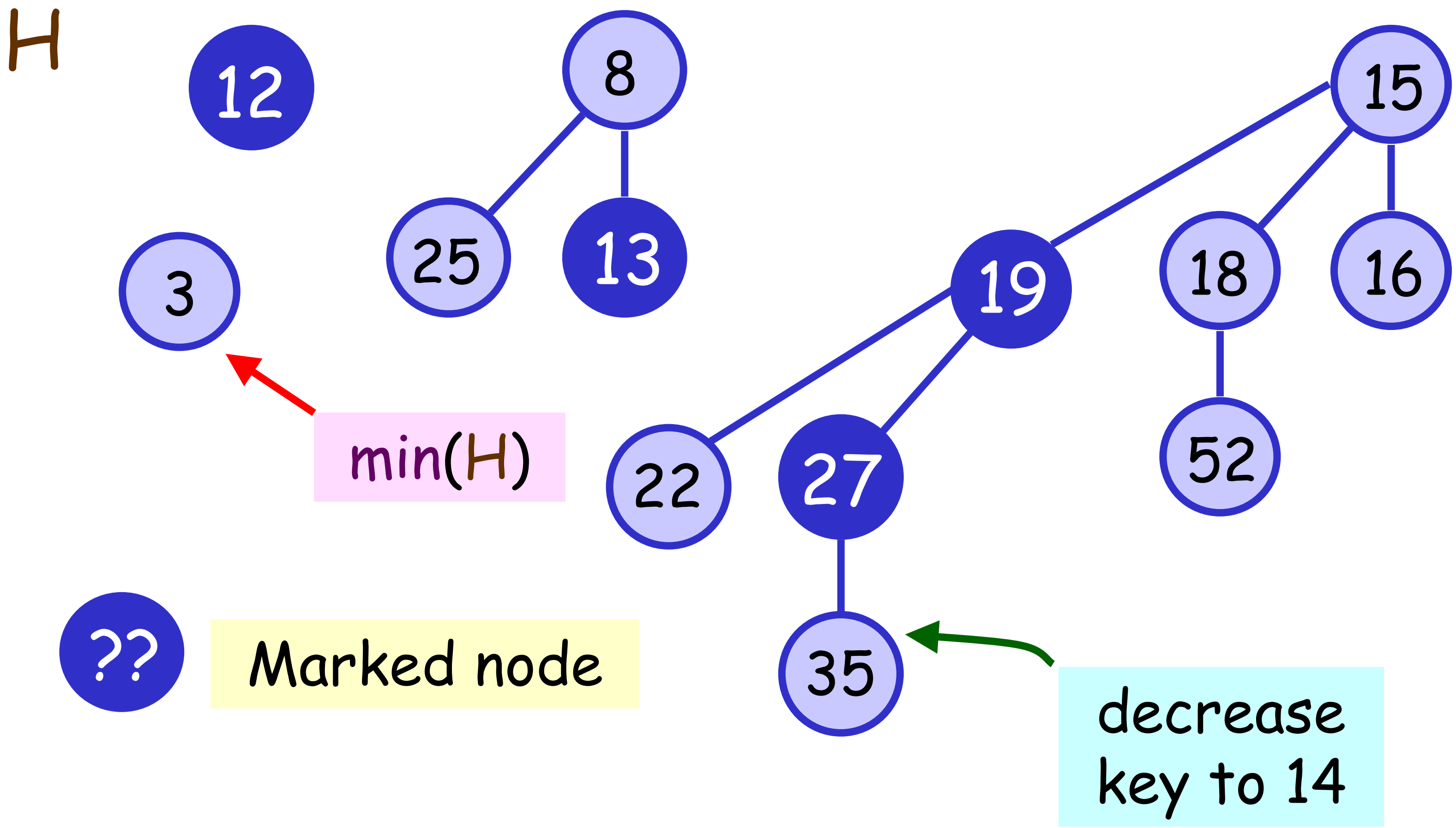
Decrease-Key (Example)

The node with key 32 has its key decreased to 18



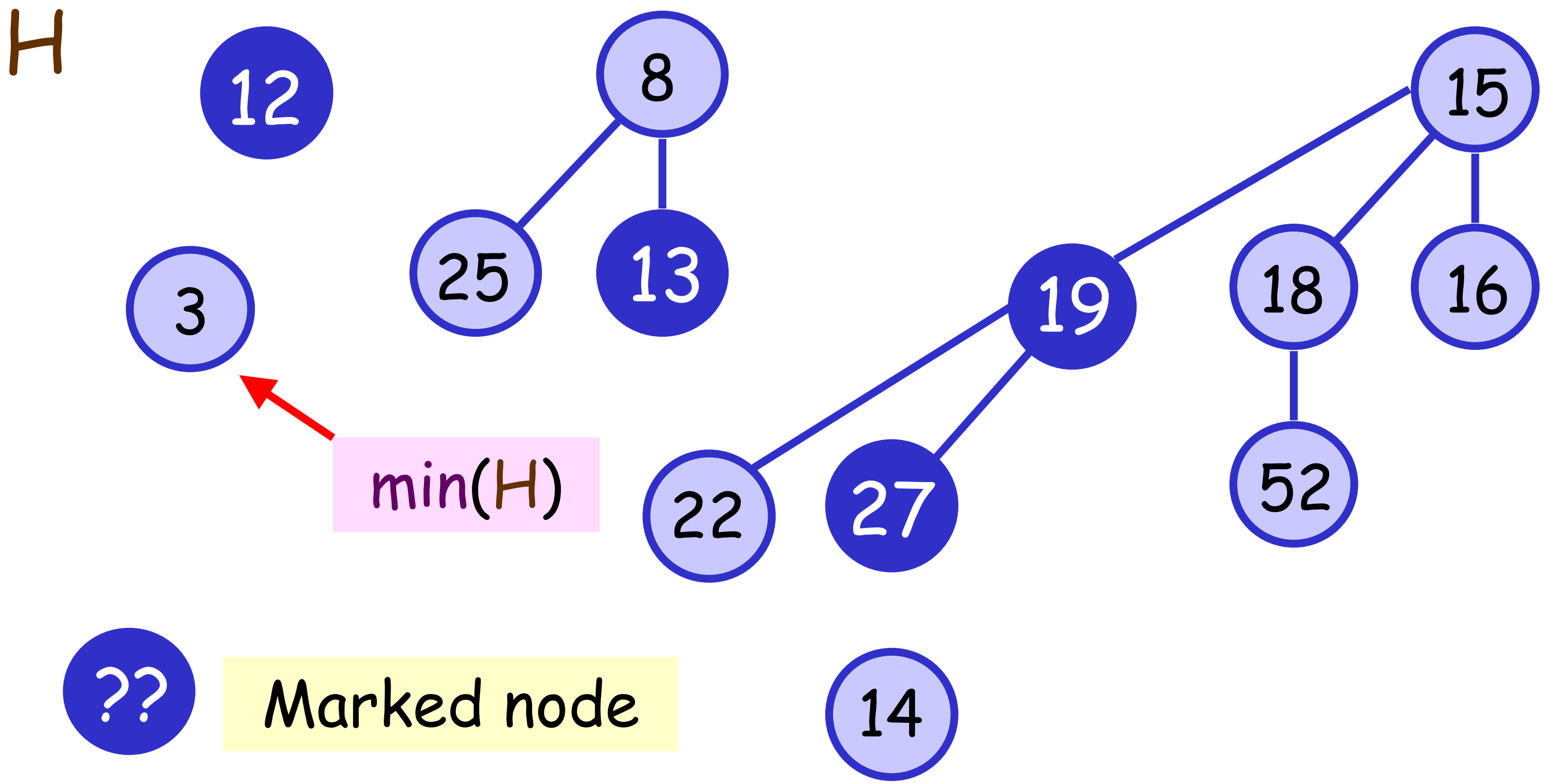
Decrease-Key (Example)

The node with key 21 has its key decreased to 3



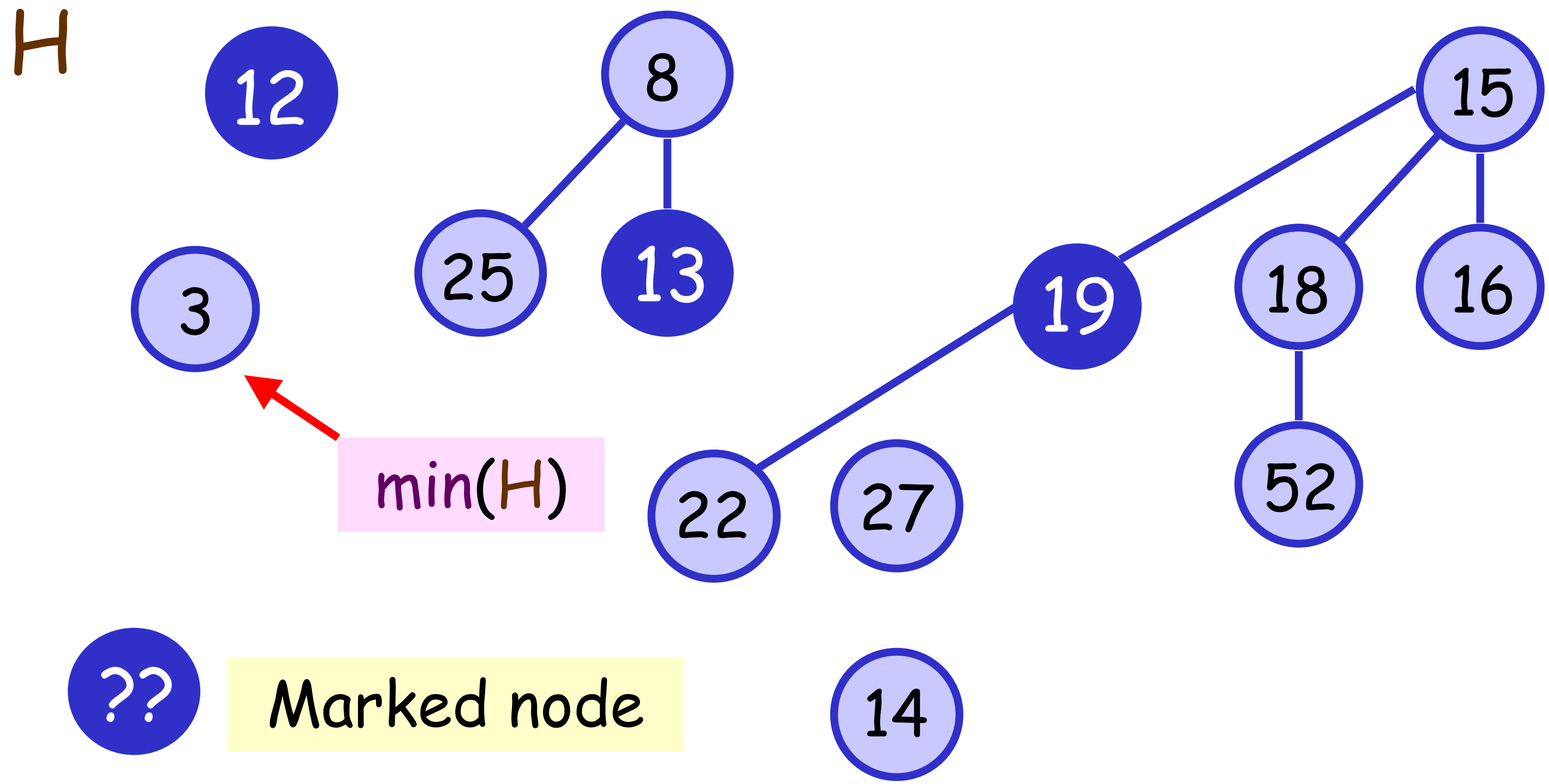
Decrease-Key (Example)

Decrease Key to 14 (Step 1)



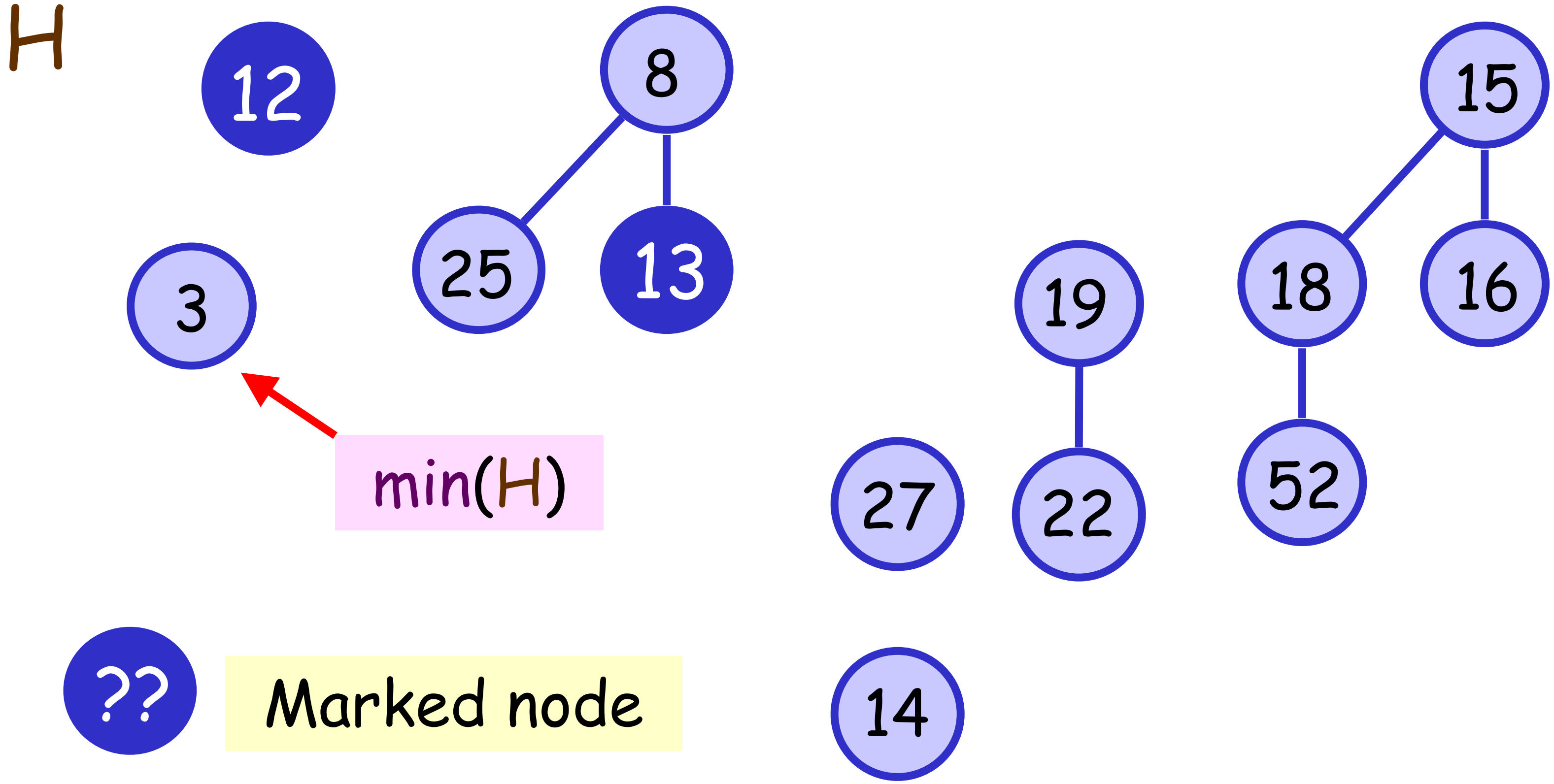
Decrease-Key (Example)

Decrease Key to 14 (Step 2)



Decrease-Key (Example)

Decrease Key to 14 (Step 3)



Decrease-Key

- We see that if Decrease-Key decides to cut a node x from its parent, it may create a series of further cuts

  ➔ we call this cascading cuts

cascading waterfall

cascading fountain

Amortized Cost

- Let H' denote the heap just before the **Decrease-Key** operation
- Let c = #cascading cuts

➔ actual cost = $O(c+1)$

potential before: $t(H') + 2m(H')$

potential after:

at most $\boxed{(t(H') + c + 1) + 2(m(H') - c + 1)}$

Why?

➔ amortized cost $\leq O(c+1) + 3 - c = O(1)$

Delete

- Delete(H, x):

  Decrease-Key(H, x, $-\infty$);

  Extract-Min(H);

→ Amortized cost

$= O(1) + O(\log n) = O(\log n)$

Bounding MaxDeg(n)

- Recall that #trees and height of a tree in a Fibonacci heap is unbounded
  - Can you obtain a component tree in Fibonacci heap whose height = $\Theta(n)$?
- In contrast, we shall show that the degree of a node is bounded by $O(\log n)$
  - We denote this bound by MaxDeg(n)

Bounding MaxDeg(n)

- For any node x, we let

  size(x) = #nodes in the subtree
  rooted at x, including itself

  deg(x) = #children of x

- Our idea is to show that size(x) is exponential in deg(x)

A Useful Lemma

Lemma: Let x be a node in the Fibonacci heap, and suppose that $\deg(x) = k$

Let $y_1, y_2, \ldots, y_k$ be the children of x, ordered by the time they are linked to x

Then, we have:

$$\deg(y_1) \geq 0, \qquad \text{and}$$

$$\deg(y_j) \geq j - 2 \quad \text{for } j = 2,3,\ldots,k$$

Proof

- $deg(y_1) \geq 0$ is trivial
- By the time y_j is linked to x, the nodes $y_1, y_2, \ldots, y_{j-1}$ were already linked to x
  ➔ x has at least $j-1$ children
  ➔ $deg(y_j)$ at that time
  $= deg(x)$ at that time $\geq j-1$
- Since then, y_j loses at most 1 child (why??), so $deg(y_j) \geq j-2$

Fibonacci Number

- We are about to see why Fibonacci heap has “Fibonacci” in its name
- Define the k^{th} Fibonacci Number, F_k, by:
  - $F_0 = 0$, $F_1 = 1$,
  - For $k \geq 2$, $F_k = F_{k-2} + F_{k-1}$
- For example, the first few Fibonacci numbers are: 0, 1, 1, 2, 3, 5, 8, 13, 21 …

Two Lemmas on F_k

Lemma: For all integers $k \geq 0$,

$$F_{k+2} = 1 + F_0 + F_1 + F_2 + \dots + F_k$$

Lemma: For all integers $k \geq 0$,

$$F_{k+2} \geq \varphi^k ,$$

where $\varphi = (1+\sqrt{5})/2 = 1.61803\dots$

How to prove? (By induction)

A Key Result

Combining previous lemmas, we can show:

Lemma: Let x be a node in the Fibonacci heap, and suppose that $\deg(x) = k$

Then, we have:

$$\text{size}(x) \geq F_{k+2} \geq \varphi^{k}$$

Proof: Let s_k = min possible size of a node whose degree is k

$\rightarrow$ $\text{size}(x) \geq s_k$

Proof of Key Result

- We shall show by induction that:

  $$s_k \geq F_{k+2}$$

  If it is true, our proof completes

- Base Case: $s_0 = 1 = F_2$ and $s_1 = 2 = F_3$
- Inductive Case:

  Assume $s_j \geq F_{j+2}$ for all $j = 0,1,\ldots, k-1$

Proof of Key Result

- Consider any deg-k node whose size = s_k
- By our lemma, we see that its children, say $y_1, y_2, \ldots, y_k$ have degrees satisfying:

$$\deg(y_1) \geq 0, \text{ and } \deg(y_j) \geq j-2 \quad \text{for } j \geq 2$$

$$\begin{aligned}
\rightarrow s_k &= 1 + \text{size}(y_1) + \text{size}(y_2) + \ldots + \text{size}(y_k) \\
&\geq 1 + s_{\deg(y_1)} + s_{\deg(y_2)} + \ldots + s_{\deg(y_k)} \\
&\geq 1 + s_0 + s_0 + \ldots + s_{k-2} \geq 1 + F_2 + F_2 + \ldots + F_k \\
&= 1 + (F_0 + F_1) + F_2 + \ldots + F_k = F_{k+2}
\end{aligned}$$

Bounding MaxDeg(n)

Immediately, we have:

Theorem: MaxDeg(n) $\leq \log_{\varphi} n = O(\log n)$

Proof: For any node x with deg k, we have:

$$n \geq \mathrm{size}(x) \geq \varphi^{k}$$

→ degree of any node $\leq \log_{\varphi} n$

→ the theorem thus follows

Remark: Since MaxDeg(n) must be an integer, we can show a tighter bound: MaxDeg(n) $\leq \lfloor \log_{\varphi} n \rfloor$