CS4311
Design and Analysis of Algorithms

Lecture 14: Amortized Analysis I

About this lecture

- Given a data structure, amortized analysis studies in a sequence of operations, the average time to perform an operation
- Introduce amortized cost of an operation

- Three Methods for the Same Purpose
  (1) Aggregate Method
  (2) Accounting Method
  (3) Potential Method

This Lecture

Super Stack

- Your friend has created a super stack, which, apart from PUSH/POP, supports:

  SUPER-POP(k): pop top k items

- Suppose SUPER-POP never pops more items than current stack size

- The time for SUPER-POP is O(k)
- The time for PUSH/POP is O(1)

Super Stack

- Suppose we start with an empty stack, and we have performed n operations
  - But we don't know the order

Questions:

- Worst-case time of a SUPER-POP ?

  Ans. $O(n)$ time   [why?]
- Total time of n operations in worst case ?

  Ans. $O(n^2)$ time   [correct, but not tight]

Super Stack

- Though we don't know the order of the operations, we still know that:
  - There are n PUSH/POP

    ➔ Time spent on PUSH/POP = O(n)
  - # items popped by all SUPER-POP cannot exceed total # items ever pushed into stack

    ➔ Time spent on SUPER-POP = O(n)

So, total time of n operations = O(n) !!!

Amortized Cost

- So far, there are no assumptions on n and the order of operations. Thus, we have:

> For any n and any sequence of n operations,
> worst-case total time = $O(n)$

- We can think of each operation performs in average $O(n) / n = O(1)$ time

→ We say amortized cost = $O(1)$ per operation (or, each runs in amortized $O(1)$ time)

Amortized Cost

- In general, we can say something like:
  - OP_1 runs in amortized $O(x)$ time
  - OP_2 runs in amortized $O(y)$ time
  - OP_3 runs in amortized $O(z)$ time

Meaning:

For any sequence of operations with
$\#OP_1 = n_1$, $\#OP_2 = n_2$, $\#OP_3 = n_3$,
worst-case total time $= O(n_1x + n_2y + n_3z)$

Binary Counter

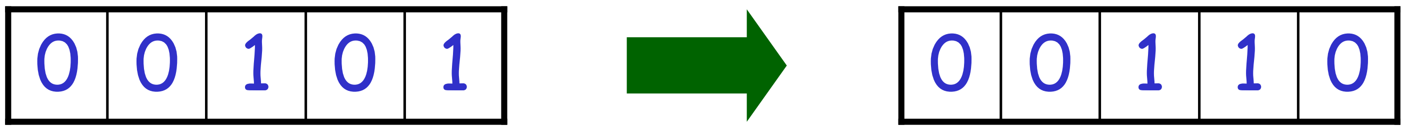
- Let us see another example of implementing a k-bit binary counter
- At the beginning, count is 0, and the counter will be like (assume k=5):

0	0	0	0	0

which is the binary representation of the count

Binary Counter

- When the counter is incremented, the content will change
- Example: content of counter when:



count = 5 cost = 2 count = 6

- The cost of the increment is equal to the number of bits flipped

Binary Counter

Special case:

When all bits in the counter is 1,
an increment resets all bits to 0

1	1	1	1	1

0	0	0	0	0

count = MAX   cost = k   count = 0

- The cost of the corresponding increment is equal to k, the number of bits flipped

Binary Counter

- Suppose we have performed n increments

Questions:

- Worst-case time of an increment ?
  Ans. $O(k)$ time

- Total time of n operations in worst case ?
  Ans. $O(nk)$ time [correct, but not tight]

Binary Counter

Let us denote the bits in the counter by

$$b_0, b_1, b_2, \ldots, b_{k-1},$$

starting from the right

b_4	b_3	b_2	b_1	b_0

Observation:

b_i is flipped only once in every 2^i increments

Precisely, b_i is flipped at x^{th} increment $\Leftrightarrow$ x is divisible by 2^i

Amortized Cost

- So, for n increments, the total cost is:

$$\sum_{i=0 \text{ to } k} \lfloor n / 2^i \rfloor$$

$$\leq \sum_{i=0 \text{ to } k} (n / 2^i) \; < \; 2n$$

- By dividing total cost with #increments,

➔ amortized cost of increment = O(1)

Aggregate Method

- The computation of amortized cost of an operation in super stack or binary counter follows similar steps:

1. Find total cost (thus, an "aggregation")
2. Divide total cost by #operations

This method is called Aggregate Method

Remarks

- In amortized analysis, the amortized cost to perform an operation is computed by the average over all performed operations
- There is a different topic called average-case analysis, which studies average performance over all inputs
- Both are useful, but they just study different things

Example: Average-Case Analysis

- Consider building a binary search tree for n numbers with random insertion order
  - Final height varies on insertion order

- Suppose each of the $n!$ possible insertion orders is equally likely to be chosen

- Then, we may be able to compute the average height of the tree
  - average is over all insertion orders

Example: Average-Case Analysis

- In fact, we can show that

  average height $= \Theta(\log n)$

  and very likely, height $= \Theta(\log n)$

- So, we can say

  average search time $= \Theta(\log n)$

- However, we cannot say

  amortized search time $= \Theta(\log n)$ ... why?