CS4311
Design and Analysis of Algorithms

Lecture 11: Dynamic Programming III

About this lecture

- We will see more examples today

Writing a Translation Program

- Suppose we want to design a program to translate English texts on food to Chinese
- First problem to solve:

Given an English word, can we quickly search for its Chinese equivalent?

- E.g., Apple → 蘋果, Banana → 香蕉,
  Pizza → 比薩, Burger → 漢堡,
  Hotdog → 熱狗, Spaghetti → 意麵

Writing a Translation Program

- However, some English words may not be common to have a Chinese equivalent
  - In this case, we report not found

- E.g., Biryani (a South Asian dish)
  Burrito (a common Mexican food)
  Jambalaya (a famous Louisiana dish)
  Okonomiyaki (a kind of Japanese pizza)

Writing a Translation Program

- Let n = # of English words in our database with Chinese equivalent

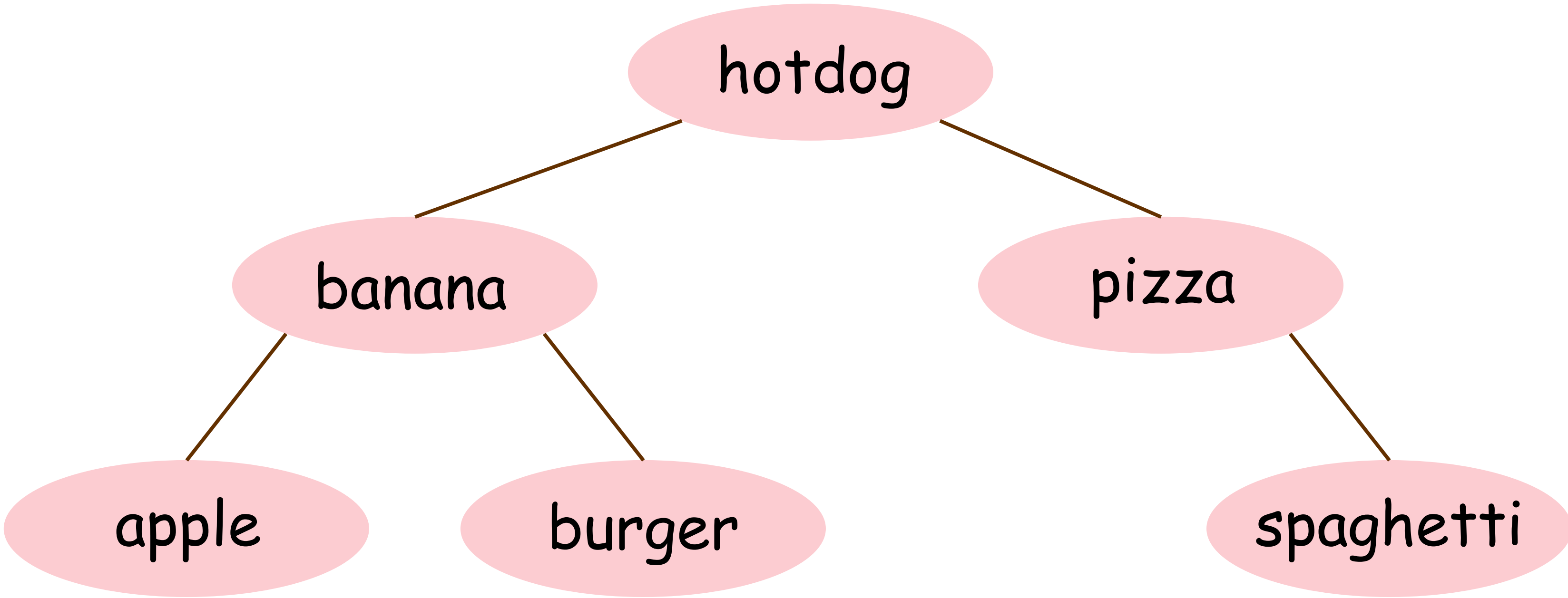
Solution 1: Hashing

- Good, but need a good hash function

Solution 2: Balanced Binary Search Tree

- worst-case $O(\log n)$ time per query

Balanced Binary Search Tree



Keys = words in the database

Writing a Translation Program

- In real life, different words do not appear with the same frequencies

  E.g., apple may be more often than pizza

- Also, there may be different frequencies for the unsuccessful searches

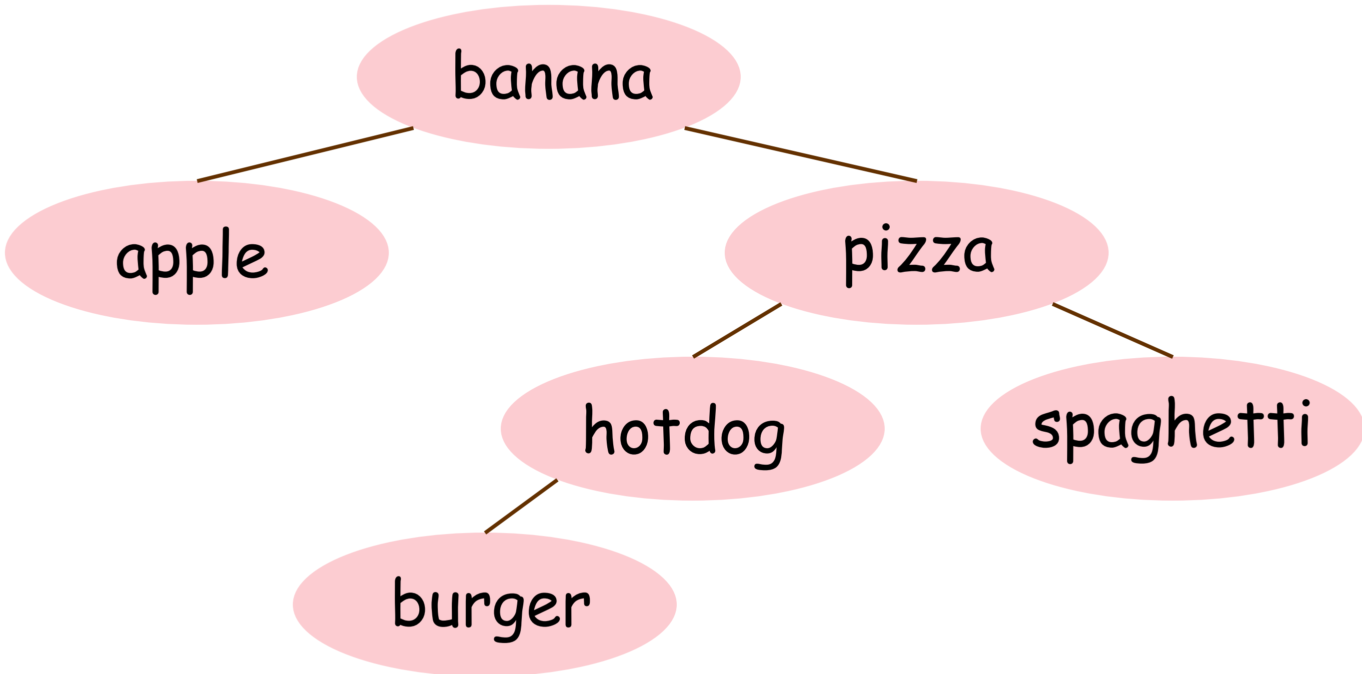
  E.g., we may unluckily search for a word in the range (hotdog, pizza) more often than in the range (spaghetti, $+\infty$)

- Suppose your friend in Google gives you probabilities of what a search will be:

< apple	0.01
= apple	0.21
(apple, banana)	0.10
= banana	0.18
(banana, burger)	0.05
= burger	0.01
(burger, hotdog)	0.12

= hotdog	0.02
(hotdog, pizza)	0.04
= pizza	0.04
(pizza, spaghetti)	0.11
= spaghetti	0.07
> spaghetti	0.04

- Given these probabilities, we may want words that are searched more frequently to be nearer the root of the search tree

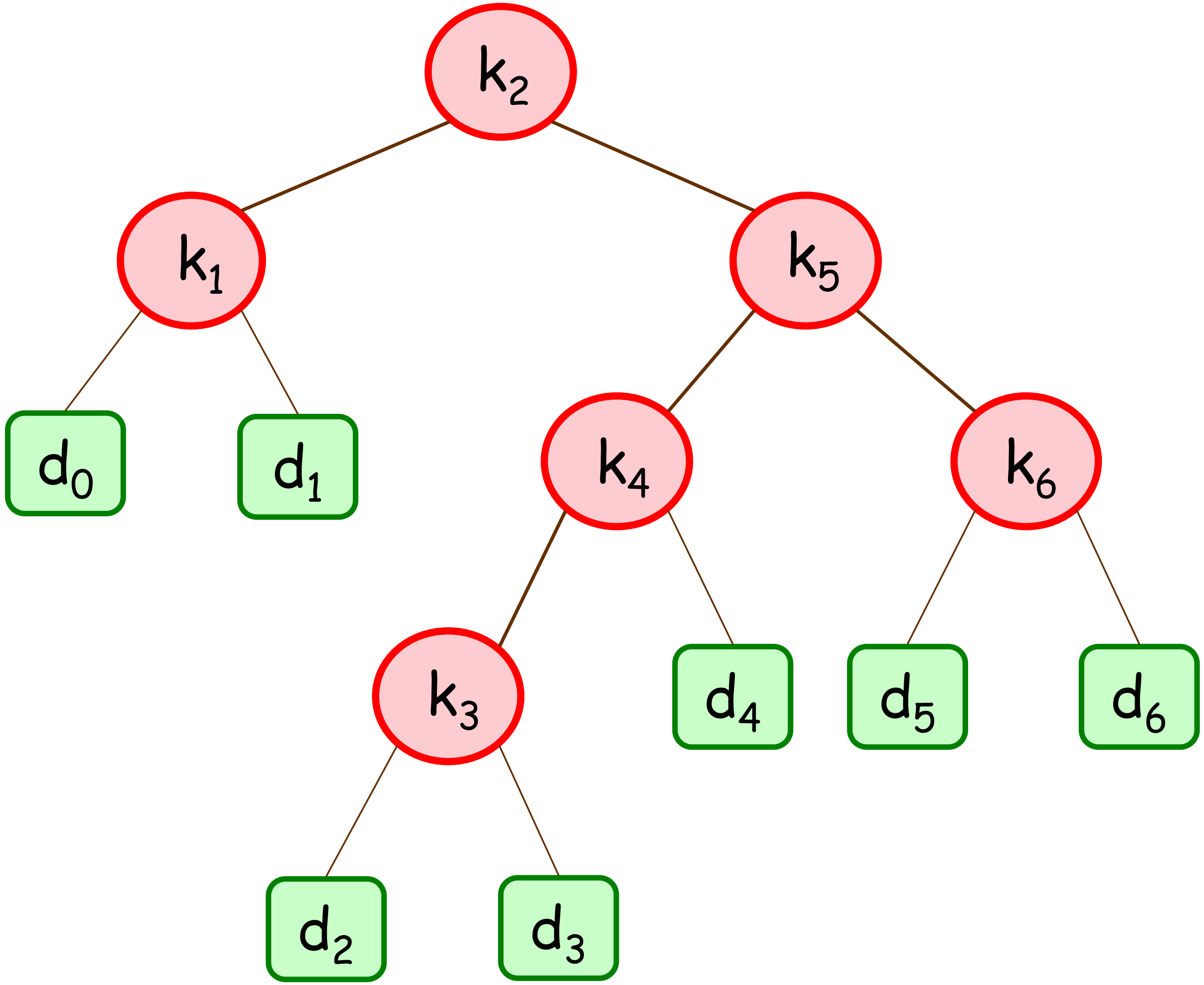


This tree has better expected performance

Expected Search Time

- We can modify the search tree slightly (by adding dummy leaves), and define the expected search time as follows:
- Let $k_1 < k_2 < \ldots < k_n$ denote the n keys, which correspond to the internal nodes
- Let $d_0 < d_1 < d_2 < \ldots < d_n$ be dummy keys for ranges of the unsuccessful search

  ➔ dummy keys correspond to leaves



Search tree of Page 9 after modification

Expected Search Time

Lemma: Based on the modified search tree:

- when we search for a word k_i,

  search time = depth(k_i) + 1

- when we search for a word in range d_j,

  search time = depth(d_j) + 1

Expected Search Time

- Let p_i = Pr(k_i is searched)
- Let q_j = Pr(word in d_j is searched)

➔ $$\sum_i p_i + \sum_j q_j = 1$$

Then, expected search time

$$= \sum_i p_i\,(\mathrm{depth}(k_i) + 1) + \sum_j q_j\,(\mathrm{depth}(d_j) + 1)$$

$$= 1 + \sum_i p_i\,\mathrm{depth}(k_i) + \sum_j q_j\,\mathrm{depth}(d_j)$$

Optimal Binary Search Tree

Question:

Given the probabilities p_i and q_j,
can we construct a binary search tree
whose expected search time is minimized?

Such a search tree is called an
Optimal Binary Search Tree

Optimal Substructure

Let T = optimal BST for the keys

$$(k_i, k_{i+1}, \ldots , k_j; d_{i-1}, d_i, \ldots , d_j).$$

Let L and R be its left and right subtrees.

Lemma: Suppose k_r is the root of T. Then,

- L must be an optimal BST for the keys
  $$(k_i , k_{i+1}, \ldots , k_{r-1}; d_{i-1}, d_i, \ldots , d_{r-1})$$
- R must be an optimal BST for the keys
  $$(k_{r+1}, k_{r+2}, \ldots , k_j; d_r, d_{r+1}, \ldots , d_j)$$

Optimal Substructure

Let $e_{i,j}$ denote expected search time within an optimal BST for the keys

$$(k_i, k_{i+1}, \ldots , k_j; d_{i-1}, d_i, \ldots , d_j)$$

→ $e_{i,i-1} = Pr(d_{i-1}) * 1 = q_{i-1}$

Let $w_{i,j}$ denote the sum of the probabilities of the keys $(k_i, k_{i+1}, \ldots , k_j; d_{i-1}, d_i, \ldots , d_j)$

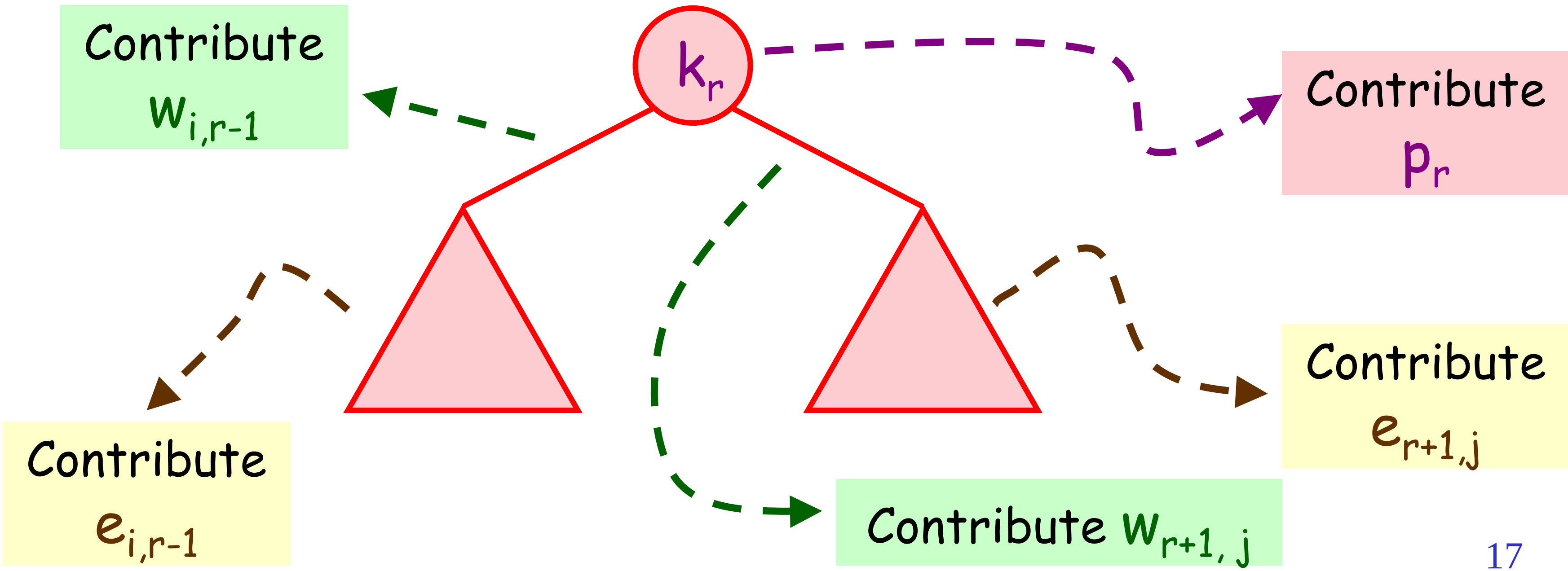
$$= \sum_{s=i \text{ to } j} p_s + \sum_{t=i-1 \text{ to } j} q_t$$

→ $w_{i,i-1} = Pr(d_{i-1}) = q_{i-1}$

Optimal Substructure

Lemma: For any $j \geq i$,

$$e_{i,j} = \min_r \{ p_r + e_{i,r-1} + w_{i,r-1} + e_{r+1,j} + w_{r+1,j}$$
$$= \min_r \{ e_{i,r-1} + e_{r+1,j} + w_{i,j} \}$$



Optimal Binary Search Tree

Define a function Compute_E(i,j) as follows:

```
Compute_E(i, j)  /* Finding e_{i,j} */
  1. if (i == j+1)  return q_j;
  2. m = ∞;
  3. for (r = i, i+1, …, j) {
       g = Compute_E(i,r-1) + Compute_E(r+1,j) + w_{i,j} ;
       if (g < m)  m = g;
     }
 4. return m ;
```

Optimal Binary Search Tree

Question: We want to get Compute_E(1,n)...
What is its running time?

- Similar to Matrix-Chain Multiplication, the recursive function runs in $\Omega(3^n)$ time
- Also it will examine at most once for all possible binary search tree

➔ Running time = O(C(2n-2,n-1)/n)

Catalan Number

Overlapping Subproblems

Here, we can see that :

To Compute_E(i,j) and Compute_E(i,j+1), there are many COMMON subproblems:

Compute_E(i,i+1), …, Compute_E(i,j-1)

So, in our recursive algorithm, there are many redundant computations !

Question: Can we avoid it ?

Bottom-Up Approach

- Let us create a 2D table E to store all $e_{i,j}$ values once they are computed
- Let us also create a 2D table W to store all $w_{i,j}$

We first compute all entries in W.

Next, we compute $e_{i,j}$ for $j-i = 0,1,2,\ldots,n-1$

Bottom-Up Approach

BottomUp_E( ) `/* Finding min #operations */`

1. Fill all entries of W
2. for j = 1, 2, …, n, set $E[j+1, j] = q_j$;
3. for (length = 0,1,2,…, n-1)

   Compute E[i,i+length] for all i;

   `// From W and E[x,y] with |x-y| < length`
4. return E[1,n] ;

Running Time = $\Theta(n^3)$

Remarks

- Again, a slight change in the algorithm allows us to get the exact structure of the optimal binary search tree

- Also, we can make minor changes to the recursive algorithm and obtain a memoized version (whose running time is $O(n^3)$)