# On Unfolding Trees and Polygons on Various Lattices

Author: Sheung-Hung Poon,

Presented by: Elena Mumford

Technical University of Eindhoven (TU/e),
The Netherlands

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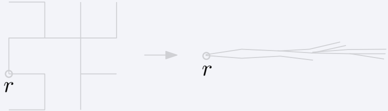
# Introduction

- Rigid edges can rotate around vertex w/o crossing other edges.
- "*Straightening*" = move rigid edges to lie on a straight line.

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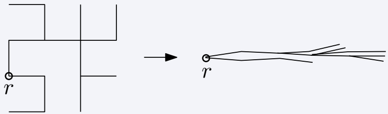


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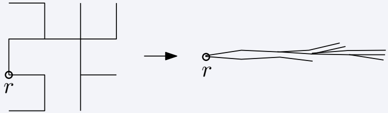


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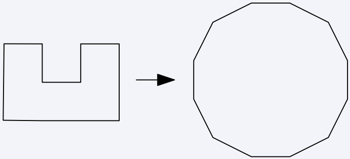

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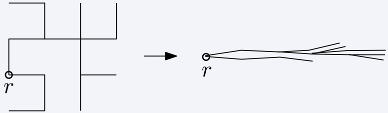


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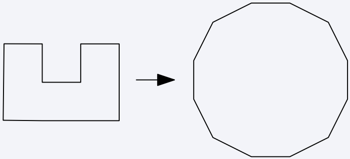

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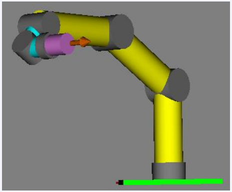
  - movements of robot arms,



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(Borrowed from Univ. College London)

  - wire bending,
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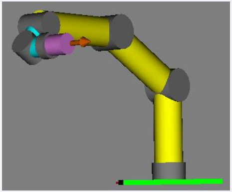

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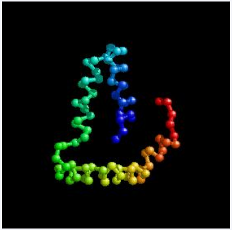

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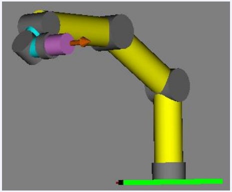

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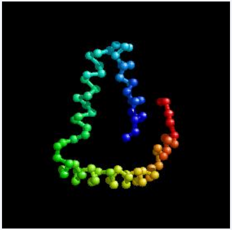

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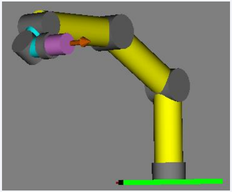

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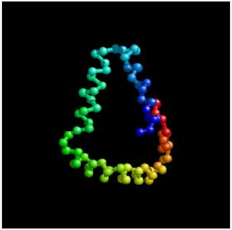

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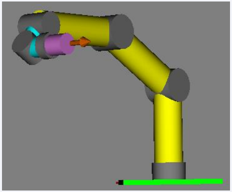

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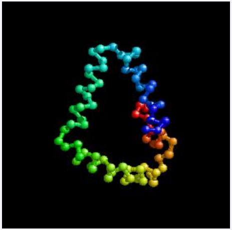

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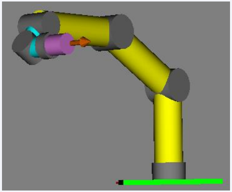

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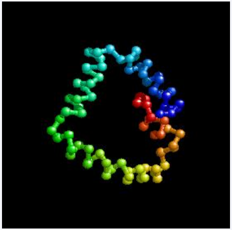

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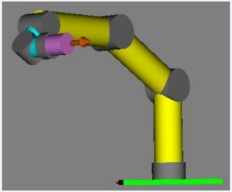

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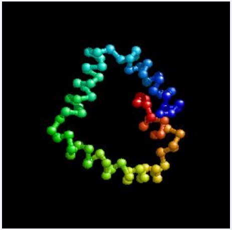

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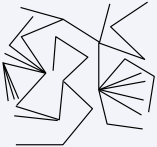
# Definition of Lattice Polygons/Trees

In this talk, we mainly consider lattice polygons/trees.

- A *unit polygon/tree* = with all its edges of unit-length.



- A *lattice polygon/tree* = with all its edges from a lattice.



Square lattice tree



Hexagonal lattice tree



Triangular lattice tree

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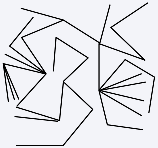

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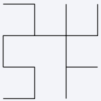
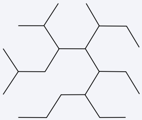
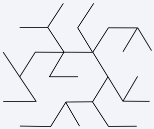


Square lattice tree



Hexagonal lattice tree



Triangular lattice tree

# Previous Work

- *Carpenter's Rule Conjecture* (solved):
  Chains (polygons) in 2D can be straightened (convexified).
  [Connelly, Demaine and Rote '00][Strienu '00]
- Trees (polygons) in 4D+ can be straightened (convexified).
  [Cocan and O'Rourke '01]
- A tree in 2D & a 5-chain in 3D can lock. [Biedl et al '01]



($a$) A locked tree in 2D.



(b) A locked 5-chain in 3D

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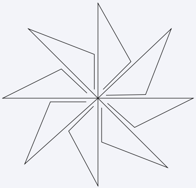

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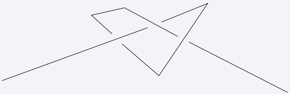

(b) A locked 5-chain in 3D

- PSPACE-complete: to reconfigure 2D-trees or 3D-chains. [Alt et al. '03]
- A unit tree *of diameter 4* can always be straightened. [Poon '05]
- A 2D/3D lattice tree can always be straightened, and
  A 2D lattice polygon can always be convexified. [Poon '06]

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# Our Results

A "*move*" = a monotonic increase/decrease of angle at a vertex.

- **Theorem**

  A hexagonal/triangular *lattice chain can be straightened in $O(n)$ moves and time ($n$ = no. of edges).*

- **Theorem**

  A hexagonal/triangular *lattice tree can be straightened in $O(n^2)$ moves and time.*

- **Theorem**

  A hexagonal/triangular *lattice polygon can be convexified in $O(n^2)$ moves and time.*

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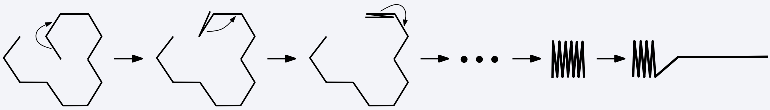
# Hexagonal Lattice Chains

**Algorithm:**

- Fold up the end edges of the chain iteratively.
- Unfold the final folded spring/zig-zag path.



PS. The algorithm is similar to that for *square lattice*.

# Hexagonal Lattice Trees

- Given a hexagonal lattice tree $P$.
- Let $r$ be root = the *leftmost* vertex of $P$.

## Algorithm:

- Our algorithm proceeds by pulling $P$ to the left successively until the whole tree is straightened.
- In each pulling step:
  - Each vertex $v$ is pulled along its edge connecting to its parent;
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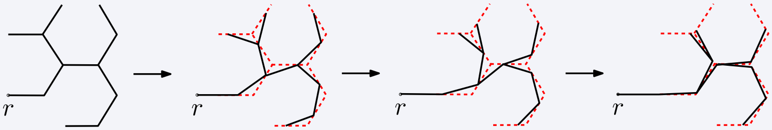
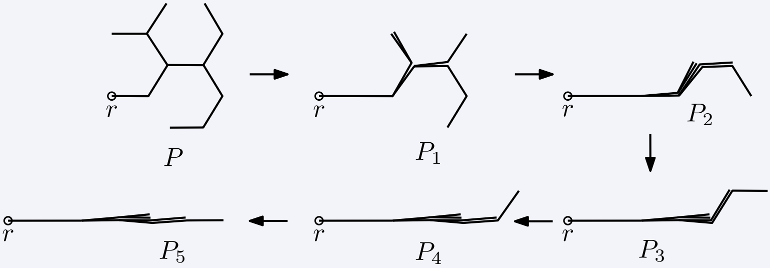


- The pulling step is repeated $n$ times so that, finally, tree $P$ is straightened.



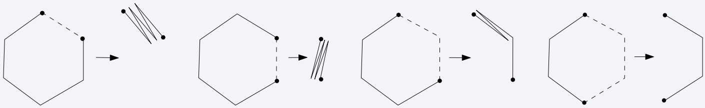

- Each pulling step takes $O(n)$ moves. Thus $n$ pulling steps take $O(n^2)$ moves in total.

# Hexagonal Lattice Polygons

- Use similar technique of *block-collapsing* as square lattice.
- New definition: A *block* = a hexagonal cell.

**Algorithm:**

Collapse leftmost collapsible block iteratively.



*Case 1$a$* *Case 1$b$* *Case 2* *Case 3*

- Observe: operation in *Case 3*, no edges are reduced.
- After at most $O(n)$ operations of *Case 3*,
  we reach one *Case 1* or *Case 2*.
- Thus whole algorithm takes $O(n^2)$ moves and time.

# Triangular Lattice Chains

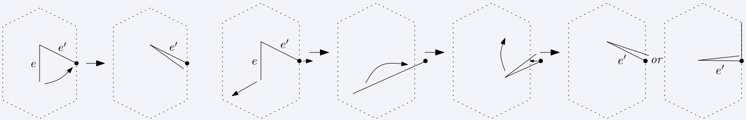
- Use similar technique as for *square & hexagonal lattice*.

**Algorithm:**

- Fold up the end edges of the chain iteratively.
- Unfold the final folded spring/zig-zag path.

Cases to handle:

1. When angle $\alpha$ % end edge & its adjacent edge is $\pi/3$:




$(a)$ $(b)$

2. When angle $\alpha = 2\pi/3$ or $\pi$, we can handle similarly.

# Triangular Lattice Trees

- Given a triangular lattice tree $P$.
- Let $r$ be root = the *leftmost* vertex of $P$.

**Algorithm:**

- Our algorithm proceeds by pulling $P$ to the left successively until the whole tree is straightened.
- In each pulling step:
  - Each vertex $v$ is pulled along its edge (or its extension) connecting to its parent;
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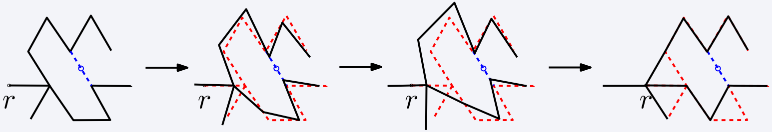



- During motion, a vertex never exceed middle point of an *extension* edge(dashed blue). So *no* edge crossings occur.

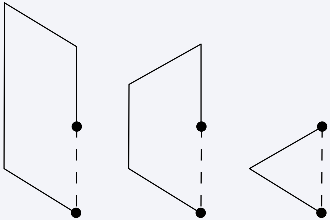
- Each pulling step takes $O(n)$ moves, and thus the whole algorithm takes $O(n^2)$ moves and time.

# Triangular Lattice Polygons

- We extend *block-collapsing* technique for square/hexagonal lattice.

## New definition:

A *block* = a $(a)$ *parallelogram,* $(b)$ *trapezoidal or* $(c)$ *triangular* block.



$(a)$ $(b)$ $(c)$

## Algorithm:

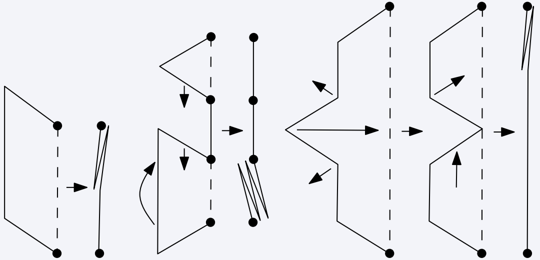
Collapse leftmost collapsible block iteratively.

Cases to handle:

(a) Collapse a parallelogram block;

(b) Collapse two trapezoidal/triangular blocks; and

(c) Collapse an extended triangular/trapezoidal block.



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- We obtain results for chains/trees/polygons on hexagonal or triangular lattice can be straightened/covexified.

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