CS5321 Numerical Optimization Homework 7

Due June 18

1. (30%) Consider the problem

$$\begin{array}{ll} \min_{x_1,x_2} & 0.1 * (x_1 - 3)^2 + x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 - 1 \leq 0 \end{array} \tag{1}$$

   (a) Write down the KKT conditions for (1).

   (b) Solve the KKT conditions and find the optimal solutions, including the Lagrangian parameters.

   (c) Compute the reduced Hessian and check the second order conditions for the solution.

2. (20%) The trust region method (for unconstrained optimization problem) needs to solve a local model in each step

$$\begin{array}{ll} \min_{\vec{p}} & m(\vec{p}) = \frac{1}{2}\vec{p}^T A\vec{p} + \vec{g}^T\vec{p} \\ \text{s.t.} & \vec{p}^T\vec{p} \leq \Delta^2. \end{array}$$

   Prove that the optimal solution $\vec{p}^*$ of the local model satisfies

$$\begin{array}{rcl} (A + \lambda I)\vec{p}^* & = & -\vec{g} \\ \lambda(\Delta - \|\vec{p}^*\|) & = & 0 \\ (A + \lambda I) & \text{is} & \text{positive semidefinite.} \end{array}$$

   (Hint: to prove the last statement, you only need to consider the directions in the *critical cone*.)

3. (50%) Implement the Interior Point Method (IPM), as shown in Figure 1, to solve linear programming problem.

$$\begin{array}{ll} \min_{\vec{x}} & \vec{c}^T\vec{x} \\ \text{s.t.} & A\vec{x} \geq \vec{b} \end{array}$$

   You can assume $\vec{x}_0 = 0$ is a feasible interior point, and $\vec{b} \leq 0$.

(1) Given $\vec{x}_0 = 0$, $\vec{\lambda}_0 = \vec{e}$, $\vec{s}_0 = -\vec{b}$, and $\mu_0 = \dfrac{\vec{\lambda}_0^T \vec{s}_0}{m}$, and choose $\sigma_k \in [0.1, 0.9]$

(2) For $k = 0, 1, \ldots$ until $\mu_k \leq 10^{-6}$.

(3) Let $\Lambda_k = \text{diag}(\vec{\lambda}_k)$ and $S_k = \text{diag}(\vec{s}_k)$. Solve

$$\begin{pmatrix} 0 & -A^T & 0 \\ -A & 0 & I \\ 0 & S_k & \Lambda_k \end{pmatrix} \begin{pmatrix} \Delta\vec{x}_k \\ \Delta\vec{\lambda}_k \\ \Delta\vec{s}_k \end{pmatrix} = \begin{pmatrix} A^T\vec{\lambda}_k - \vec{c} \\ \vec{b} - A\vec{x}_k - \vec{s}_k \\ \sigma_k\mu_k\vec{e} - \Lambda_k S_k \vec{e} \end{pmatrix},$$

(4) Compute α_k such that

$$(\vec{x}_{k+1}, \vec{\lambda}_{k+1}, \vec{s}_{k+1}) = (\vec{x}_k, \vec{\lambda}_k, \vec{s}_k) + \alpha_k(\Delta\vec{x}_k, \Delta\vec{\lambda}_k, \Delta\vec{s}_k)$$

is in the region $N(\gamma) = \{(\vec{x}, \vec{\lambda}, \vec{s}) | \lambda_i s_i \geq \gamma\mu_k, \forall i = 1, 2, \ldots, n\}$ for $\gamma = 10^{-3}$.

(5) Update $\vec{x}_{k+1}, \vec{\lambda}_{k+1}, \vec{s}_{k+1}$, and $\mu_{k+1} = \dfrac{\vec{\lambda}_{k+1}^T \vec{s}_{k+1}}{m}$.

End For

Figure 1: The interior point method for solving linear programming