CS5321 Numerical Optimization Homework 3

Line Search and Multi-dimensional Problems

Due April 5, 2012

1. (40%) Recall the Wolfe conditions for step length α

   - Sufficient decrease cond: $f(x_k + \alpha p_k) \leq f(x_k) + c_1 \alpha f'(x_k) p_k$,
   - Curvature condition: $f'(x_k + \alpha p_k) p_k \geq c_2 f'(x_k) p_k$,

   for $0 < c_1 < c_2 < 1$ and $p_k = \pm 1$. Suppose function f is *Lipschitz continuous*, which means

   $$|f'(x) - f'(y)| \leq L|x - y| \text{ for all } x, y \text{ and } L > 0.$$

   Prove the following results.

   (a) The curvature condition implies $\alpha \geq \dfrac{(c_2 - 1) f'(x_k) p_k}{L} > 0$. In other word, the curvature condition enforces α not too small.

   (b) Let $x_{k+1} = x_k + \alpha_k p_k$. The sufficient decrease condition, with the result of (a), implies

   $$f(x_k) - f(x_{k+1}) \geq \frac{c_1(1 - c_2)}{L} |f'(x_k)|^2 > 0.$$

   This means the sufficient decrease condition ensures the *sufficient decrease* of function value.

   (c) Suppose f is bounded below, which means $f(x_0) - f(x_k)$ is smaller than some positive constant for all k. Show

   $$\sum_{k=0}^{\infty} |f'(x_k)|^2 < \infty.$$

2. (60%) Let $f(x, y) = \frac{1}{2}(x - 1)^2 + \frac{9}{2}(y - 1)^2$. This is a positive definite quadratic with minimizer at $(x^*, y^*) = (1, 1)$.

(a) Derive the gradient g and the Hessian H of f.

(b) Write Matlab codes to implement the steepest descent method and Newton's method with $\vec{x}_0 = (9, 0)$, and compare their convergent results. The formula of the steepest descent method is

$$\vec{x}_{k+1} = \vec{x}_k - \frac{\vec{g}_k^T \vec{g}_k}{\vec{g}_k^T H_k \vec{g}_k} \vec{g}_k,$$

and the formula of Newton's method is

$$\vec{x}_{k+1} = \vec{x}_k - H_k^{-1} \vec{g}_k,$$

where $\vec{g}_k = g(\vec{x}_k)$ and $H_k = H(\vec{x}_k)$.

*** In Matlab, use `x=A\b` to compute $x = A^{-1}b$.

(c) Draw the trace of $\{\vec{x}_k\}$ for the steepest descent method and Newton's method. An example code for trace drawing is given as follows.

```
function draw_trace()
   % draw the contour of the function z = ((x-1)^2+9*(y-1)^2)/2;
   step = 0.1;
   X = 0:step:9;
   Y = -1:step:1;
   n = size(X,2);
   m = size(Y,2);
   Z = zeros(m,n);
   for i = 1:n
      for j = 1:m
         Z(j,i) = f(X(i),Y(j));
      end
   end
   contour(X,Y,Z,100)

   % function definition
       function z = f(x,y)
           z = 1/2*(x-1)^2+9/2*(y-1)^2;
       end
   end
```