CS5321 Numerical Optimization Homework 1

Due March 10

1. (30%) For a single variable function $f(x)$, its derivative represents the slop of its tangent. But the concept of slope is only for two dimensional lines. Actually, the geometrical meaning of $f'(x)$ is also associated with the *normal vector* if we treat the $(x, f(x))$ as a curve on a two-dimensional space. The normal vector, or simply called the "normal," to a surface (curve) is a vector perpendicular to its tangent plane (tangent line).

   (a) Prove that the normal vector at $(x_1, f(x_1))$ is $(f'(x_1), -1)$.

   Suppose the normal vector is (a, b). Let (x, y) be any point on the line, except $(x_1, f(x_1))$. The vector $(x - x_1, y - f(x_1))$ is perpendicular with the normal vector,

   $$\langle (a, b), (x - x_1, y - f(x_1)) \rangle = a(x - x_1) + b(y - f(x_1)) = 0.$$

   Since all points one the line also satisfy the line equation $y = f'(x_1)(x - x_1) + f(x_1)$,

   $$f'(x_1)(x - x_1) - (y - f(x_1)) = 0$$

   $(f'(x_1), -1)$ is parallel to the normal vector.

   (b) For a multivariable function $f(\vec{x})$, prove the normal vector at $(\vec{x}_1, f(\vec{x}_1))$ is $(\nabla f(\vec{x}_1)^T, -1)$, where $\nabla f(\vec{x}_1)$ is the gradient of f at $\vec{x}_1$.

   Suppose the normal vector of $f(\vec{x})$ is $(\vec{a}, b)$, which is perpendicular with its tangent plane. Let $(\vec{x}, y)$ be any point on the tangent plane, except $(\vec{x}_1, f(\vec{x}_1))$. The vector $(\vec{x} - \vec{x}_1, y - f(\vec{x}_1))$ is perpendicular with the normal vector,

   $$\langle (\vec{a}, b), (\vec{x} - \vec{x}_1, y - f(\vec{x}_1)) \rangle = \vec{a}^T(\vec{x} - \vec{x}_1) + b(y - f(\vec{x}_1)) = 0.$$

   Since this surface is a function of $\vec{x}$, $b \neq 0$, we can set $b = -1$ and the plan equation becomes $y = \vec{a}^T(\vec{x} - \vec{x}_1) + f(\vec{x}_1)$.

Let x_i be the ith element of $\vec{x}_1$ and a_i be the ith element of $\vec{a}$. To get a_i, we compute the directional derivative along the ith axis: $d(f, \vec{e}_i) = \partial f / \partial x_i$. Therefore, $\vec{a} = \nabla f(\vec{x}_1)$.

2. (70%) Consider the function $f(x) = \sin(5x) + x^2 - 3x$ for $x \in [0, 2]$

   (a) Plot the function and print it.

   (b) Write the Taylor expansion of $f(x)$ at $x = 1$ to degree 3 and use Matlab to evaluate the difference of $f(1.5)$ and the Taylor expansion of $f(x)$ at $x = 1.5$ for different degrees.

   (c) Use Matlab to implement (1) the bisection method and (2) Newton's method to find the root of $f(x)$ to the precision 10^{-6} and compare their convergence. What initial guesses of Newton's method do you use? Will different initial guesses make different convergent results.

You may find some Matlab's sample codes in the last year's class website. `http://www.cs.nthu.edu.tw/~cherung/teaching/2010cs5321/homework/hw1.pdf`