Numerical Optimization

Unit 7: Constrained Optimization Problems

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Problem formulation

General formulation

$$\begin{aligned} \min_{\vec{x}} \quad & f(\vec{x}) \\ \text{s.t.} \quad & c_i(\vec{x}) = 0, \quad i \in \mathcal{E} \\ & c_i(\vec{x}) \geq 0, \quad i \in \mathcal{I}. \end{aligned} \tag{1}$$

- $\mathcal{E}$ is the index set for equality constraints; $\mathcal{I}$ is the index set for inequality constraints.
- $\Omega = \{\vec{x} | c_i(\vec{x}) = 0, i \in \mathcal{E}$ and $c_j(\vec{x}) \geq 0, j \in \mathcal{I}\}$ is the set of feasible solutions.
- The function $f(\vec{x})$ and $c_i(\vec{x})$ can be linear or nonlinear.

Example 1

Example

$$\min_{x_1, x_2} f(x_1, x_2) = x_1 + x_2$$

$$\text{s.t. } c(x_1, x_2) = x_1^2 + x_2^2 - 2 = 0.$$

- The optimal solution is at $\vec{x}^* = (x_1^*, x_2^*) = (-1, -1)$
- The gradient of c is $\nabla c = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$, and $\nabla c(\vec{x}^*) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$
- The gradient of $\nabla f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Properties of the optimal solution in Example 1

1. $f(\vec{x}^* + \vec{s}) \geq f(\vec{x}^*)$ for small enough $\vec{s}$. (why?)

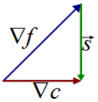
$$f(\vec{x}^*+\vec{s}) = f(\vec{x}^*)+\nabla f(\vec{x}^*)^T\vec{s}+O(\|\vec{s}\|^2) \Rightarrow \nabla f(\vec{x}^*)^T\vec{s} \geq 0, \quad \forall \vec{s}, \|\vec{s}\| \leq \epsilon$$

2. $\vec{c}(\vec{x}^*) = \vec{c}(\vec{x}^* + \vec{s}) = 0$ for small enough $\vec{s}$. (why?)

$$\vec{c}(\vec{x}^* + \vec{s}) \approx c(\vec{x}^*) + \nabla c(\vec{x}^*)^T\vec{s} = 0 \Rightarrow \nabla c(\vec{x}^*)^T\vec{s} = 0, \quad \forall \vec{s}, \|\vec{s}\| \leq \epsilon$$

3. From 1. and 2., we can infer that ∇f must be parallel to ∇c. (why?)

If ∇f is not parallel to ∇c, there will be an $\vec{s}$ that makes $\nabla f^T\vec{s} < 0$ and $\nabla c^T\vec{s} = 0$, as shown in the figure.



Example 2

Example

$$\min_{x_1, x_2} f(x_1, x_2) = x_1 + x_2$$

$$\text{s.t. } c(\vec{x}) = 2 - x_1^2 - x_2^2 \geq 0$$

What are the properties of the optimal solution in Example 2?

1. If $f(\vec{x}^*)$ is inside the circle , then $\nabla f(\vec{x}^*) = 0$. (why?)
2. If $f(\vec{x}^*)$ is on the circle , then $c(\vec{x}^*) = 0$, which goes back to the equality constraint.
3. From 1. and 2., we can conclude that $\nabla f(\vec{x}^*) = \lambda \nabla c(\vec{x}^*)$ for some scalar λ.
    - In the first case, $\lambda = 0$.
    - In the second case, λ is the scaling factor of $\nabla f(\vec{x}^*)$ and $\nabla c(\vec{x}^*)$.

Lagrangian function

The Lagrangian function

$$\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) - \lambda c(\vec{x}) \tag{2}$$

- $\nabla_{\vec{x}} \mathcal{L} = \dfrac{\partial \mathcal{L}}{\partial \vec{x}} = \nabla f(\vec{x}) - \lambda \nabla c(\vec{x})$.
- $\nabla_{\lambda} \mathcal{L} = \dfrac{\partial \mathcal{L}}{\partial \lambda} = -c(\vec{x})$.
- Therefore, at the optimal solution , $\nabla \mathcal{L} = \begin{pmatrix} \nabla_{\vec{x}} \mathcal{L}(\vec{x}^*) \\ \nabla_{\lambda} \mathcal{L}(\vec{x}^*) \end{pmatrix} = 0$.
- If $c(\vec{x}^*)$ is inactive , $\lambda^* = 0$. $\Rightarrow$ The complementarity condition $\lambda^* c(\vec{x}^*) = 0$.
- The scalar λ is called *Lagrange multiplier*.

Example 3

Example

$$\min_{x_1,x_2} \quad f(x_1,x_2) = x_1 + x_2$$
$$\text{s.t.} \quad c_1(\vec{x}) = 2 - x_1^2 - x_2^2 \geq 0$$
$$c_2(\vec{x}) = x_2 \geq 0$$

- $\nabla c_1 = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix}$, $\nabla c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\nabla f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- The optimal solution $\vec{x}* = (-\sqrt{2}, 0)^T$, at which $\nabla c_1(\vec{x}^*) = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$.
- $\nabla f(\vec{x}^*)$ is a linear combination of $\nabla c_1(\vec{x}^*)$ and $\nabla c_2(\vec{x}^*)$.

Example 3

- For this example, the Lagrangian $\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \lambda_1 c_1(\vec{x}) - \lambda_2 c_2(\vec{x})$, and

$$\nabla \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = \begin{pmatrix} \nabla_{\vec{x}} \mathcal{L} \\ \nabla_{\lambda_1} \mathcal{L} \\ \nabla_{\lambda_2} \mathcal{L} \end{pmatrix} = \begin{pmatrix} \nabla f(\vec{x}^*) - c_1(\vec{x})/2\sqrt{2} - c_2(\vec{x}) \\ -c_1(\vec{x}^*) \\ -c_2(\vec{x}^*) \end{pmatrix} = \vec{0}.$$

  - What is $\vec{\lambda}^*$?
  - The examples suggests the first order necessity condition for constrained optimizations is the gradient of the Lagrangian is zero. But is it true?

Example 4

Example

$$\min_{x_1,x_2} \quad f(x_1, x_2) = x_1 + x_2$$
$$\text{s.t.} \quad c_1(\vec{x}) = (x_1^2 + x_2^2 - 2)^2 = 0$$

- $\nabla f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\nabla \vec{c}(\vec{x}) = \begin{pmatrix} 4(x_1^2 + x_2^2 - 2)x_1 \\ 4(x_1^2 + x_2^2 - 2)x_2 \end{pmatrix}$.
- Optimal solution is $(-1, -1)$, but $\nabla c(-1, -1) = (0, 0)^T$ is not parallel to ∇f.

Example 5

Example

$$\min_{x_1,x_2} \quad f(x_1,x_2) = x_1 + x_2$$
$$\text{s.t.} \quad c_1(\vec{x}) = 1 - x_1^2 - (x_2-1)^2 \geq 0$$
$$c_2(\vec{x}) = -x_2 \geq 0$$

- $\nabla c_1 = \begin{pmatrix} -2x_1 \\ -2(x_2-1) \end{pmatrix}$, $\nabla c_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and $\nabla f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- The only solution is $(0,0)$. $\nabla c_1(0,0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\nabla c_2(0,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.
- At the optimal solution, ∇f is not a linear combination of ∇c_1 and ∇c_2.

Regularity conditions

Regularity conditions: conditions of the constraints

Linear independence constraint qualifications (LICQ)

Given a point $\vec{x}$ and its active set $\mathcal{A}(\vec{x})$, LICQ holds if the gradients of the constraints in $\mathcal{A}(\vec{x})$ are linearly independent.

KKT conditions

KKT conditions: the first order necessary condition for the COP

The KKT conditions(Karush-Kuhn-Tucker)

Suppose $\vec{x}^*$ is a solution to the problem defined in (1), where f and c_i are continuously differentiable and the LICQ holds at $\vec{x}^*$. Then there exist a lagrangian multiplier vector $\vec{\lambda}^*$ s.t. the following conditions are satisfied at $(\vec{x}^*, \vec{\lambda}^*)$

1. $\nabla_{\vec{x}^*} \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = 0$
2. $c_i(\vec{x}^*) = 0 \quad \forall i \in \mathcal{E}$
3. $c_i(\vec{x}^*) \geq 0 \quad \forall i \in \mathcal{I}$
4. $\lambda_i^* c_i(\vec{x}^*) \geq 0$ (Strict complementarity condition: either $\lambda_i^* = 0$ or $c_i(\vec{x}^*) = 0$.)
5. $\lambda_i^* \geq 0, \forall i \in \mathcal{I}$ ($\lambda_i^* > 0, \forall i \in \mathcal{I} \cup \mathcal{A}^*$ if the strict complementarity condition holds.)

Two definitions for the proof of KKT

Tangent cone

A vector $\vec{d}$ is said to be a *tangent* to a point set Ω at point $\vec{x}$ if there are a sequence $\{\vec{z}_k\}$ and a sequence $\{t_k\}$, in which $t_k > 0$ and $\{t_k\}$ converges to 0, such that

$$\lim_{k\to\infty} \frac{\vec{z}_k - \vec{d}}{t_k} = \vec{d}.$$

The set of all tangents to Ω at $\vec{x}^*$ is called the *tangent cone*.

The set of linearized feasible directions

Given a feasible point $\vec{x}$ and the active constraint set $\mathcal{A}(\vec{x})$, the set of linearized feasible directions is defined as

$$\mathcal{F}(\vec{x}) = \left\{ \vec{d} \;\middle|\; \begin{array}{ll} \vec{d}^T \nabla c_i(\vec{x}) = 0 & \forall i \in \mathcal{E}, \\ \vec{d}^T \nabla c_i(\vec{x}) \geq 0 & \forall i \in \mathcal{A}(\vec{x}) \cap \mathcal{I} \end{array} \right\}.$$

It can be shown that $\mathcal{F}(\vec{x})$ is a cone.

Outline of the proof of the KKT conditions

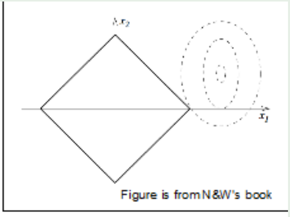
1. $\forall \vec{d} \in$ tangent cone at $\vec{x}^*$ $\vec{d}^T \nabla f \geq 0$. (Using the idea of tangent cone to prove it)
2. Tangent cone at $\vec{x}^*$ = feasible directions at $\vec{x}^*$
3. By 1 and 2 , $\vec{d}^T \nabla f \geq 0$ for $\forall \vec{d} \in F(\vec{x}^*)$
4. By Farkas lemma , either one need be true.[1]
   (a) $\exists \vec{d} \in \mathbb{R}^n$, $\vec{d}^T \nabla f < 0$, $B^T \vec{d} \geq 0$ $\vec{c}^T \vec{d} = 0$
   (b) $\nabla f \in \{By + Cw | y \geq 0\}$
5. Since (a) is not true (Because of 3) , (b) must be true.

[1] The proof of Farkas lemma can be found in last year's homework 4.

Example 6

Example

$$\begin{array}{ll} \min_{x_1,x_2} & (x_1 - \frac{3}{2})^2 + (x_2 - \frac{1}{2})^4 \\ \text{s.t.} & c_1(\vec{x}) = 1 - x_1 - x_2 \geq 0 \\ & c_2(\vec{x}) = 1 - x_1 + x_2 \geq 0 \\ & c_3(\vec{x}) = 1 + x_1 - x_2 \geq 0 \\ & c_4(\vec{x}) = 1 + x_1 + x_2 \geq 0 \end{array}$$



$$\nabla c_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \nabla c_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \nabla c_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \nabla c_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\vec{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \nabla f(\vec{x}^*) = \begin{pmatrix} 2(x_1^* - \frac{3}{2}) \\ 4(x_2^* - \frac{1}{2})^3 \end{pmatrix} = 11 \begin{pmatrix} -1 \\ -1/2 \end{pmatrix}.$$

$$\vec{\lambda}^* = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{pmatrix}^T$$

The second order condition

- With constraints, we don't need to consider all the directions. The directions we only need to worried about are the "feasible directions".
- The critical cone $\mathcal{C}(\vec{x}^*, \vec{\lambda}^*)$ is a set of directions defined at the optimal solution $(\vec{x}^*, \vec{\lambda}^*)$

$$\vec{w} \in \mathcal{C}(\vec{x}^*, \vec{\lambda}^*) \Leftrightarrow \begin{cases} \nabla c_i(\vec{x}^*)^T \vec{w} = 0 & \forall i \in \mathcal{E} \\ \nabla c_i(\vec{x}^*)^T \vec{w} = 0 & \forall i \in \mathcal{A}(\vec{x}^*) \cap \mathcal{I},\ \lambda_i^* > 0 \\ \nabla c_i(\vec{x}^*)^T \vec{w} \geq 0 & \forall i \in \mathcal{A}(\vec{x}^*) \cap \mathcal{I},\ \lambda_i^* = 0 \end{cases}$$

The second order necessary condition

Suppose $\vec{x}^*$ is a local minimizer at which the LICQ holds, and $\vec{\lambda}^*$ is the Lagrange multiplier. Then $\vec{w}^T \nabla^2_{xx} \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) \vec{w} \geq 0, \quad \forall \vec{w} \in \mathcal{C}(\vec{x}^*, \vec{\lambda}^*)$.

Proof

We perform Taylor expansion at $\vec{x}^*$ and evaluate its neighbor $\vec{z}$,

$$\begin{aligned}\mathcal{L}(\vec{z}, \vec{\lambda}^*) =& \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) + (\vec{z} - \vec{x}^*)^T \nabla_x \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) \\ &+ \frac{1}{2}(\vec{z} - \vec{x}^*)^T \nabla^2_{xx} \mathcal{L}(\vec{x}^*, \vec{\lambda}^*)(\vec{z} - \vec{x}^*) + O(\|\vec{z} - \vec{x}^*\|^3)\end{aligned}$$

Since $\mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = f(\vec{x}^*)$ (why?) and $\nabla_x \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = 0$. Let $\vec{w} = \vec{z} - \vec{x}^*$, which is in the critical cone.

$$\begin{aligned}\mathcal{L}(\vec{z}, \vec{\lambda}^*) &= f(\vec{z}) - \sum_{\forall\, i} \lambda_i^* c_i(\vec{z}) \\ &= f(\vec{z}) - \sum_{\forall\, i} \vec{\lambda}_i^* (c_i(\vec{x}^*) + \nabla c_i(\vec{x}^*)^T \vec{w}) = f(\vec{z})\end{aligned}$$

Thus, $f(\vec{z}) = \mathcal{L}(\vec{z}, \vec{\lambda}^*) = f(\vec{x}^*) + \frac{1}{2}\vec{w}^T \nabla^2_{xx} \mathcal{L}(\vec{x}^*, \vec{\lambda}^*)\vec{w} + O(\|\vec{z} - \vec{x}^*\|^3)$, which is larger than $f(\vec{x}^*)$ if $\vec{w}^T \nabla^2_{xx} \mathcal{L}(\vec{x}^*, \vec{\lambda}^*)\vec{w} \geq 0$.

Example

Example

$$\min_{x_1,x_2} -0.1(x_1-4)^2 + x_2^2 \text{ s.t. } x_1^2 - x_2^2 - 1 \geq 0.$$

$$\mathcal{L}(\vec{x},\lambda) = -0.1(x_1-4)^2 + x_2^2 + \lambda(x_1^2 - x_2^2 - 1)$$

$$\nabla_x \mathcal{L} = \begin{pmatrix} -0.2(x_1-4) + 2\lambda x_1 \\ 2x_2 - 2\lambda x_2 \end{pmatrix}, \nabla_{xx}\mathcal{L} = \begin{pmatrix} -0.2 - 2\lambda_1 & 0 \\ 0 & 2-2\lambda \end{pmatrix}$$

at $\vec{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \lambda^* = 0.3 \quad$ $\nabla C(\vec{x}^*) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

The critical cone $\mathcal{C}(\vec{x}^*) = \left\{ \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \middle| w_2 \in \mathbb{R} \right\}$

$$\nabla_{xx}\mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = \begin{pmatrix} -0.4 & 0 \\ 0 & 1.4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & w_2 \end{pmatrix} \begin{pmatrix} -0.4 & 0 \\ 0 & 1.4 \end{pmatrix} \begin{pmatrix} 0 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & 1.4w_2 \end{pmatrix} \begin{pmatrix} 0 \\ w_2 \end{pmatrix} = 1.4w_2^2 > 0$$

Some easy way to check the condition

Is there any easy way to check the condition?

- Let Z be a matrix whose column vectors span the subspace of $\mathcal{C}(\vec{x}^*, \vec{\lambda}^*)$

$$\Rightarrow \begin{cases} \forall \vec{w} \in \mathcal{C}(\vec{x}^*, \vec{\lambda}^*), & \exists \vec{u} \in \mathbb{R}^m \quad s.t.\ \vec{w} = Z\vec{u} \\ \forall \vec{u} \in \mathbb{R}^m, & Z\vec{u} \in \mathcal{C}(\vec{x}^*, \vec{\lambda}^*) \end{cases}$$

- To check $\vec{w}^T \nabla_{xx} \mathcal{L}^* \vec{w} \geq 0$, $\Leftrightarrow \vec{u}^T Z^T \nabla_{xx} \mathcal{L}^* Z \vec{u} \geq 0$ for all $\vec{u}$ $\Leftrightarrow Z^T \nabla_{xx} \mathcal{L}^* Z$ is positive semidefinite.
- The matrix $Z^T \nabla_{xx} \mathcal{L}^* Z$ is called the *projected Hessian*.

Active constraint matrix

- Let $A(\vec{x}^*)$ be the matrix whose rows are the gradient of the active constraints at the optimal solution $\vec{x}^*$.

$$A(\vec{x}^*)^T = [\nabla c_i(\vec{x}^*)]_{i \in \mathcal{A}(\vec{x}^*)}$$

- The critical cone $\mathcal{C}(\vec{x}^*, \vec{\lambda}^*)$ is the null space of $A(\vec{x}^*)$

$$\vec{w} \in \mathcal{C}(\vec{x}^*, \vec{\lambda}^*) \Leftrightarrow A(\vec{x}^*)\vec{w} = 0$$

- We don't consider the case that $\lambda^* = 0$ for active c_i. (Strict complementarity condition.)

Compute the null space of $A(\vec{x}^*)$

- Using QR factorization

$$A(\vec{x}^*)^T = Q \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1$$

$A \in \mathbb{R}^{m\times n},\ Q \in \mathbb{R}^{n\times n},\ R \in \mathbb{R}^{m\times m},\ Q_1 \in \mathbb{R}^{n\times m},\ Q_2 \in \mathbb{R}^{n\times (n-m)}$

- The null space of A is spanned by Q_2, which means any vectors in the null space of A is a unique linearly combination of Q_2's column vectors.

$$\vec{z} = Q_2\vec{v} \qquad A\vec{z} = R^T Q_1^T Q_2 \vec{v} = 0$$

To check the second order condition is to check if $Q_2^T \nabla^2 \mathcal{L}^* Q_2$ is positive definite.

Duality

Consider the problem: $\min_{\vec{x}\in\mathbb{R}^n} f(\vec{x})$ subject to $c(\vec{x}) = \begin{pmatrix} c_1(\vec{x}) \\ c_2(\vec{x}) \\ \vdots \\ c_m(\vec{x}) \end{pmatrix} \geq 0$

Its Lagrangian function is

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \vec{\lambda}^T c(\vec{x})$$

The dual problem is defined as

$$\max_{\vec{\lambda}\in\mathbb{R}^n} q(\vec{\lambda}) \quad \text{s.t.} \quad \vec{\lambda} \geq 0$$

where $q(\vec{\lambda}) = \inf_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda})$.

Duality

- Infimum is the global minimum of $\mathcal{L}(\cdot, \lambda)$, which may not be defined or difficult to compute.
- For f and $-c_i$ are convex, $\mathcal{L}$ is also convex $\Rightarrow$ the local minimizer is the global minimize.
- Wolfe's duality: another formulation of duality when function is differentiable.

$$\max \mathcal{L}(\vec{x}, \vec{\lambda})$$
$$s.t.\ \nabla_x \mathcal{L}(\vec{x}, \vec{\lambda}) = 0,\ \lambda \geq 0$$

Example

Example

$\min_{(x_1\ x_2)} 0.5(x_1^2 + x_2^2) \quad s.t.\ x_1 - 1 \geq 0$

- $\mathcal{L}(X_1\ X_2, \lambda) = 0.5(x_1^2 + x_2^2) - \lambda_1(x_1 - 1)$,
- $\nabla_x \mathcal{L} = \begin{pmatrix} x_1 - \lambda_1 \\ x_2 \end{pmatrix} = 0$, which implies $x_1 = \lambda_1$ and $x_2 = 0$.
- $q(\lambda) = \mathcal{L}(\lambda_1, 0, \lambda_1) = -0.5\lambda_1^2 + \lambda_1$.
- The dual problem is

$$\max_{\lambda_1 \geq 0} -0.5\lambda_1^2 + \lambda_1$$

Weak duality

Weak duality: For any $\vec{x}$ and $\vec{\lambda}$ feasible, $q(\vec{\lambda}) \leq f(\vec{x})$
$q(\lambda) = \inf_{\vec{x}}(f(\vec{x}) - \vec{\lambda}^T c(\vec{x})) \leq f(\vec{x}) - \vec{\lambda}^T c(\vec{x}) \leq f(\vec{x})$

Example

$$\min_{\vec{x}} \vec{c}^T \vec{x} \quad s.t.\ A\vec{x} - \vec{b} \geq 0,\ \vec{x} \geq 0$$

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = \vec{c}^T \vec{x} - \vec{\lambda}^T (A\vec{x} - \vec{b}) = (\vec{c}^T - \vec{\lambda}^T A)\vec{x} + \vec{b}^T \vec{\lambda}$$

Since $\vec{x} \geq 0$, if $(\vec{c} - A^T \vec{\lambda})^T < 0$, $\inf_{\vec{x}} \mathcal{L} \to -\infty$. We require $\vec{c}^T - A^T \lambda > 0$.

$$q(\vec{\lambda}) = \inf_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) = \vec{b}^T \vec{\lambda}$$

The dual problem becomes

$$\max_{\lambda} \vec{b}^T \vec{\lambda} \qquad s.t. \quad A^T \vec{\lambda} \leq 0 \ \text{ and } \vec{\lambda} \geq 0.$$

The rock-paper-scissors game (two person zero sum game)

The payoff matrix $A =$

opp \ you	Rock	Paper	Scissors
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

- Suppose the opponent's strategy is $\vec{x} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$.
- What should your strategy be to maximize the payoff?

Problem formulation

- Let $\vec{y} = (y_1, y_2)^T$. We can express this problem as

$$\max_{\vec{y}} \vec{x}^T A\vec{y} = \max_{\vec{y}} \frac{-1}{2} y_1 + \frac{1}{2} y_2$$

  Therefore, to maximize your wining chance, you should throw paper.
- On the other hand, the problem of your opponent is

$$\min_{\vec{x}} \vec{x}^T A\vec{y}$$

- What if you do not know your opponent's strategy? It becomes a min-max or max-min problem.

$$\max_{\vec{y}} \min_{\vec{x}} \vec{x}^T A\vec{y}$$

Two examples

Example

Consider the payoff matrix $A = \begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix}$, and $\vec{x}, \vec{y} \in \{0, 1\}$.

- $\min_i \max_j a_{ij} = \min_i \left\{ \max_j a_{1,j}, \max_j a_{2,j} \right\} = \min\{2, 4\} = 2.$
- $\max_j \min_i a_{ij} = \max_j \left\{ \min_i a_{i,1}, \min_i a_{i,2} \right\} = \max\{-1, 2\} = 2.$

Example

Consider the payoff matrix $A = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}$

- $\min_i \max_j a_{ij} = \min_i \left\{ \max_j a_{1,j}, \max_j a_{2,j} \right\} = \min\{2, 4\} = 2.$
- $\max_j \min_i a_{ij} = \max_j \left\{ \min_i a_{i,1}, \min_i a_{i,2} \right\} = \max\{-1, 1\} = 1.$

Strong duality theorem

Strong duality theorem

$\max_{\vec{y}} \min_{\vec{x}} F(\vec{x}, \vec{y}) = \min_{\vec{x}} \max_{\vec{y}} F(\vec{x}, \vec{y})$ if and only if there exists a point $(\vec{x}^*, \vec{y}^*)$ such that $F(\vec{x}^*, \vec{y}) \leq F(\vec{x}^*, \vec{y}^*) \leq F(\vec{x}, \vec{y}^*)$.

- Point $(\vec{x}^*, \vec{y}^*)$ is called a saddle point.