Advanced Numerical Methods	January 5, 2012

Lecture Notes 10: 3.5 & 3.6 Eigenvalue problems

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1 Symmetric Eigenvalue problems

- Bisection method
- Singular value decomposition (SVD)

1.1 Bisection method

Suppose $z_1 < z_2$, the number of eigenvalues of A in the interval $[z_1, z_2)$ equals to (number of negative eigenvalues of $(A - z_2 I)$) - (number of negative eigenvalues of $(A - z_1 I)$)

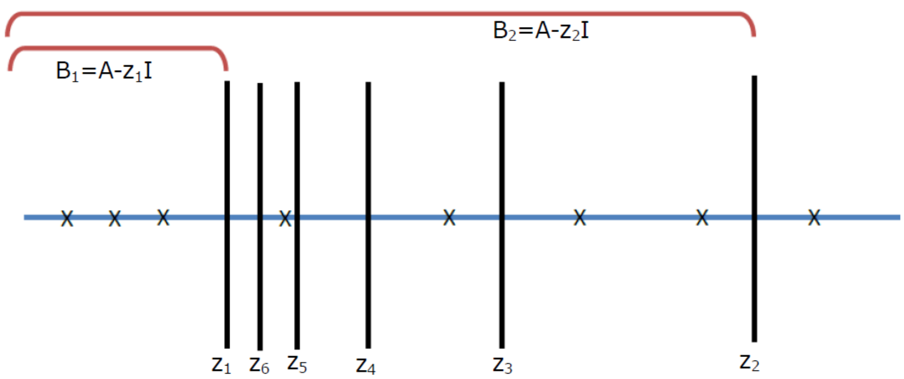


Figure 1: Bisection method

Question : $B = A - 3I$, what's the property between A and B's eigenvector and eigenvalue ?
$\Rightarrow$ B's eigenvector = A's eigenvector
$\Rightarrow$ B's eigenvalue = A's eigenvalue make a left shift of 3

Theorem : LDLT decomposition :

1. When A is symmetric, one can decompose $A = LDL^T$
   $< proof >$

$$\begin{aligned} A &= LU = LD(D^{-1}U), \text{ where } D = diag(U) \\ A^T &= (D^{-1}U)^T D^T L^T = A, \text{ where } L = (D^{-1}U)^T \\ \Rightarrow A &= LDL^T \end{aligned}$$

Algorithm 1 Bisection method $(A,\ a,\ b,\ \varepsilon)$

n_a = number of negative eigenvalues of $(A - aI)$
n_b = number of negative eigenvalues of $(A - bI)$
if $(n_a = n_b)$
    stop
$enque(a\ , n_a,\ b,\ n_b)$
while queue is not empty
    $deque(low,\ n_l,\ up,\ n_u)$
    if $(n_u == n_l)$
        stop
    else if $(up - low < \varepsilon)$
        report $eigenvalue = \frac{up+low}{2}$
    else
        $mid = \frac{up+low}{2}$
        n_m=number of negative eigenvalues of $(A - mid * I)$
        $enque(low,\ n_l,\ mid,\ n_m)$
        $enque(mid,\ n_m,\ up,\ n_u)$
    end if
end while

Note : In Cholesky decomposition, matrix A has to be both symmetric and positive definite.
However, in LDLT decomposition, matrix A only has to be symmetric.

2. $Inertia(A) = Inertia(LDL^T)$
   As long as L is nonsingular $\Rightarrow$ $Inertia(A) = Inertia(D)$

   $< proof >$ Suppose exists B such that $B = Y^{-1}AY$, A and B are similiar.

$$\begin{aligned} A &= X\Lambda X^{-1} \\ \Rightarrow B &= Y^{-1}AY = Y^{-1}X\Lambda X^{-1}Y = Z\Lambda Z^{-1} \end{aligned}$$

3. Suppose A is symmetric tridiagonal

$$\begin{aligned} A - zI &= \begin{bmatrix} a_1 - z & b_1 & & \\ b_1 & a_2 - z & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n - z \end{bmatrix} \\ &= LDL^T \\ &= \begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & \ddots & \ddots & \\ & & l_{n-1} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & l_1 & & \\ & 1 & \ddots & \\ & & \ddots & l_{n-1} \\ & & & 1 \end{bmatrix} \\ where\ d_i &= (a_i - z) - \frac{b_{i-1}^2}{d_{i-1}} \end{aligned}$$

Question : What if the d_i have zeros?
$\Rightarrow$ Then z is A's eigenvalue.

Question : How to calculate Inertia(D)?
$\Rightarrow$ From the above, we can calculate how many d_i that is positive, negative or zero.

Example : $D \in \mathbb{R}^{5\times 5}$

$$D = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & d_4 & \\ & & & & d_5 \end{bmatrix}$$

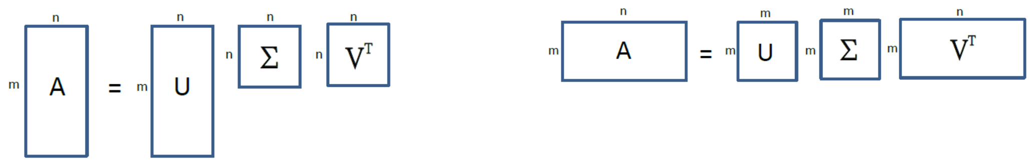
where $d_1, d_2 > 0,\ d_3, d_4 < 0,\ d_5 = 0 \Rightarrow\ Inertia(D) = (2,\ 1,\ 2)$

1.2 Singular value decomposition (SVD)

$A \in \mathbb{R}^{m\times n}$, and $m > n$, there exist orthogonal matrix U and V such that $A = U\Sigma V^T$,

where $\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}$ with $\sigma_1 \geq \sigma_1 \geq \cdots \geq \sigma_n \geq 0$, $U \in \mathbb{R}^{m\times n}$ and $V \in \mathbb{R}^{n\times n}$

$U = (\vec{u_1},\ \vec{u_2},\ \cdots,\ \vec{u_n})$, $V = (\vec{v_1},\ \vec{v_2},\ \cdots,\ \vec{v_n})$, and $\sigma_1,\ \sigma_1,\ \cdots,\ \sigma_n$ are called the singular values of A



(a) $m > n$

(b) $m < n$

Figure 2: Singular value decomposition

1. $A\vec{v_i} = \sigma_i \vec{u_i}$

   $< proof >$

$$\begin{aligned} AV &= U\Sigma(V^TV) \\ &= U\Sigma \\ \Rightarrow (A\vec{v_1},\ A\vec{v_2},\ \cdots,\ A\vec{v_n}) &= (\sigma_1\vec{u_1},\ \sigma_2\vec{u_2},\ \cdots,\ \sigma_n\vec{u_n}) \end{aligned}$$

2. $A^TA\vec{v_i} = \sigma_i^2\vec{v_i}$

   $AA^T\vec{u_i} = \sigma_i^2\vec{u_i}$

   $< proof >$ $\quad A = U\Sigma V^T, \quad A^T = V\Sigma U^T$

$$\begin{aligned} A^TA &= V\Sigma U^T U\Sigma V^T \\ &= V\Sigma^2V^T \\ &= V\Sigma^2V^{-1}, \text{ since } A^TA \text{ is symmetric} \\ \Rightarrow A^TA\vec{v_i} &= \sigma_i^2\vec{v_i} \\ &= \begin{pmatrix} \sigma_1^2\vec{v_1} & & \\ & \ddots & \\ & & \sigma_n^2\vec{v_n} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
AA^T &= U\Sigma V^T V\Sigma U^T \\
&= U\Sigma^2 U^T \\
&= \left(U \mid U' \right) \left(\begin{array}{ccc|ccc}
\sigma_1^2 & & & & & \\
& \ddots & (n) & & & \\
& (n) & \sigma_n^2 & & & \\
\hline
& & & 0 & (m-n) & \\
& & & (m-n) & \ddots & \\
& & & & & 0
\end{array} \right) \left(\frac{U^T}{U'^T} \right)
\end{aligned}
$$

Note : A^TA and AA^T are both symmetric

$< proof >$

$(A^TA)^T = A^T(A^T)^T = A^TA$

$(AA^T)^T = (A^T)^TA^T = AA^T$

Note : $A^TA \in \mathbb{R}^{n\times n}$, the Σ of A^TA is $(\sigma_1^2, \cdots, \sigma_n^2)$
However, since $AA^T \in \mathbb{R}^{m\times m}$, the Σ of AA^T is $(\sigma_1^2, \cdots, \sigma_n^2, 0, \cdots, 0)$ which has (m-n) zeros.

3. $\|A\|_2 = \sigma_1$

Definition : For vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\|\vec{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$, $p = 1, 2, \cdots, \infty$
For matrix A, $\|A\|_p = \max_{\|\vec{x}\|_p=1} \|A\vec{x}\|_p$

$< proof >$

$$
\begin{aligned}
\|\vec{x}\|_2^2 &= \vec{x}^T\vec{x} \\
\max\|A\vec{x}\|^2 &= \max(A\vec{x})^T(A\vec{x}) \\
&= \max \vec{x}^TA^TA\vec{x} \\
&= \max \frac{\vec{x}^TA^TA\vec{x}}{\vec{x}^T\vec{x}}, \text{ where } \|\vec{x}\|_p = 1 \\
&= \sigma_1^2
\end{aligned}
$$

Example : $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$

4. Calculate Singular value decomposition (using givens rotation)

$$A = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}$$

$$\Rightarrow U_1 A = \left(\begin{array}{cccc} x & x & x & x \\ x & x & x & x \\ \hline \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \hline \end{array}\right), \ U_1 \text{ is the rotation matrix of column 3 and 4.}$$

$$\Rightarrow U_2 U_1 A = \left(\begin{array}{cccc} x & x & x & x \\ \hline \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \hline 0 & x & x & x \end{array}\right), \ U_2 \text{ is the rotation matrix of column 2 and 3.}$$

$$\Rightarrow U_3 U_2 U_1 A = \left(\begin{array}{cccc} \hline \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \hline 0 & x & x & x \\ 0 & x & x & x \end{array}\right), \ U_3 \text{ is the rotation matrix of column 1 and 2.}$$

$$\Rightarrow U_3 U_2 U_1 A V_4 = \left(\begin{array}{cc|cc|} x & x & \mathbf{x} & \mathbf{0} \\ 0 & x & \mathbf{x} & \mathbf{x} \\ 0 & x & \mathbf{x} & \mathbf{x} \\ 0 & x & \mathbf{x} & \mathbf{x} \end{array}\right), \ V_4 \text{ is the rotation matrix of row 3 and 4.}$$

$$\Rightarrow U_3 U_2 U_1 A V_4 V_5 = \left(\begin{array}{c|cc|c} x & \mathbf{x} & \mathbf{0} & 0 \\ 0 & \mathbf{x} & \mathbf{x} & x \\ 0 & \mathbf{x} & \mathbf{x} & x \\ 0 & \mathbf{x} & \mathbf{x} & x \end{array}\right), \ V_5 \text{ is the rotation matrix of row 2 and 3.}$$

$$\Rightarrow U_6 U_3 U_2 U_1 A V_4 V_5 = \left(\begin{array}{cccc} x & x & 0 & 0 \\ 0 & x & x & x \\ \hline \mathbf{0} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{x} \\ \hline \end{array}\right), \ U_6 \text{ is the rotation matrix of column 3 and 4.}$$

$$\Rightarrow U_7 U_6 U_3 U_2 U_1 A V_4 V_5 = \left(\begin{array}{cccc} x & x & 0 & 0 \\ \hline \mathbf{0} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{x} \\ \hline 0 & 0 & x & x \end{array}\right), \ U_7 \text{ is the rotation matrix of column 2 and 3.}$$

$$\Rightarrow U_7 U_6 U_3 U_2 U_1 A V_4 V_5 V_8 = \left(\begin{array}{cc|cc|} x & x & \mathbf{0} & \mathbf{0} \\ 0 & x & \mathbf{x} & \mathbf{0} \\ 0 & 0 & \mathbf{x} & \mathbf{x} \\ 0 & 0 & \mathbf{x} & \mathbf{x} \end{array}\right), \ V_8 \text{ is the rotation matrix of row 3 and 4.}$$

$$\Rightarrow U_9 U_7 U_6 U_3 U_2 U_1 A V_4 V_5 V_8 = \left(\begin{array}{cccc} x & x & 0 & 0 \\ 0 & x & x & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{x} \\ \hline \end{array}\right), \ U_9 \text{ is the rotation matrix of column 3 and 4.}$$

$$= U A V^T = C = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ 0 & a_2 & b_2 & 0 \\ 0 & 0 & a_3 & b_3 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

$$
\begin{aligned}
B &= C^T C \\
\Rightarrow G_1^T B G_1 &= G_1^T C^T C G_1, \text{ where } G_1 \text{ is the rotation matrix of column 1 and 2 in matrix } B \\
&= \underbrace{G_1^T C^T}_{B'} \underbrace{(C G_1)}_{B'^T}
\end{aligned}
$$

Note : Since $G_1^T C^T$ and CG_1 are mutually transposed, we only have to look one part and the other part is its transpose.

$$
\begin{aligned}
CG_1 &= \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & x & 0 \\ \hline 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} \left(\begin{array}{cc|cc} cos & -sin & 0 & 0 \\ sin & cos & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
&= \begin{pmatrix} x & x & 0 & 0 \\ \color{red}{x} & x & x & 0 \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} \text{, we try to transform it to a bidiagonal form} \\
\Rightarrow G_2 C G_1 &= \begin{pmatrix} \mathbf{x} & \mathbf{x} & \color{red}{\mathbf{x}} & \mathbf{0} \\ \color{red}{\mathbf{0}} & \mathbf{x} & \mathbf{x} & \mathbf{0} \\ \hline 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} \text{, } G_2 \text{ is the rotation matrix of row 1 and 2.} \\
\Rightarrow G_2 C G_1 G_3 &= \left(\begin{array}{c|cc|c} x & \mathbf{x} & \color{red}{\mathbf{0}} & 0 \\ 0 & \mathbf{x} & \mathbf{x} & 0 \\ 0 & \color{red}{\mathbf{x}} & \mathbf{x} & x \\ 0 & \mathbf{0} & \mathbf{0} & x \end{array}\right) \text{, } G_3 \text{ is the rotation matrix of column 2 and 3.} \\
\Rightarrow G_4 G_2 C G_1 G_3 &= \left(\begin{array}{cccc} x & x & 0 & 0 \\ \hline \mathbf{0} & \mathbf{x} & \mathbf{x} & \color{red}{\mathbf{x}} \\ \mathbf{0} & \color{red}{\mathbf{0}} & \mathbf{x} & \mathbf{x} \\ \hline 0 & 0 & 0 & x \end{array}\right) \text{, } G_4 \text{ is the rotation matrix of row 2 and 3.} \\
\Rightarrow G_4 G_2 C G_1 G_3 G_5 &= \left(\begin{array}{cc|cc|} x & x & \mathbf{0} & \mathbf{0} \\ 0 & x & \mathbf{x} & \color{red}{\mathbf{0}} \\ 0 & 0 & \mathbf{x} & \mathbf{x} \\ 0 & 0 & \color{red}{\mathbf{x}} & \mathbf{x} \end{array}\right) \text{, } G_5 \text{ is the rotation matrix of column 3 and 4.} \\
\Rightarrow G_6 G_4 G_2 C G_1 G_3 G_5 &= \underbrace{\left(\begin{array}{cccc} x & x & 0 & 0 \\ 0 & x & x & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \mathbf{0} & \color{red}{\mathbf{0}} & \mathbf{x} \\ \hline \end{array}\right)}_{bidiagonal} \text{, } G_6 \text{ is the rotation matrix of row 3 and 4.} \\
&= UCV^T
\end{aligned}
$$

$$
\begin{aligned}
B &= C^T C \\
&= \begin{pmatrix} a_1 & 0 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & b_2 & a_3 & 0 \\ 0 & 0 & b_3 & a_4 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ 0 & a_2 & b_2 & 0 \\ 0 & 0 & a_3 & b_3 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} a_1^2 & a_1 b_1 & 0 & 0 \\ a_1 b_1 & a_2^2 + b_1^2 & a_2 b_2 & 0 \\ 0 & a_2 b_2 & a_3^2 + b_2^2 & a_3 b_3 \\ 0 & 0 & a_3 b_3 & a_4^2 + b_3^2 \end{pmatrix}}_{tridiagonal}
\end{aligned}
$$

Note : Since we have run so many QR decompositions,

the $U_\infty \underbrace{C_\infty}_{\Sigma} V_\infty^T$ in the end will become $\begin{pmatrix} x & x \to 0 & 0 & 0 \\ 0 & x & x \to 0 & 0 \\ 0 & 0 & x & x \to 0 \\ 0 & 0 & 0 & x \end{pmatrix}$