Advanced Numerical Methods December 22, 2011

Lecture Notes 6: Givens roration

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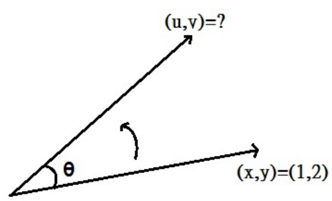
1 Givens roration

- Example: Given $(x, y) = (1, 2)$, $\theta = \frac{\pi}{3}$, $(u, v) =?$



Let $a = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})$, $b = x + yi$, $c = u + vi$
$\Rightarrow c = a \cdot b = \cos(\frac{\pi}{3})x - \sin(\frac{\pi}{3})y + (\sin(\frac{\pi}{3})x + \cos(\frac{\pi}{3})y)i$
Then we can know the rotation matrix from (x,y) to (u,v),

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Example: Given $\begin{pmatrix} x \\ y \end{pmatrix}$and $\begin{pmatrix} u \\ v \end{pmatrix}$, $x^2 + y^2 = u^2 + v^2$, compute the rotation matrix.

Let $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{z} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\cos\angle(\vec{w}, \vec{z}) = \frac{\vec{w}^T\vec{z}}{\|\vec{w}\|\|\vec{z}\|} = \frac{xv + yu}{x^2 + y^2}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

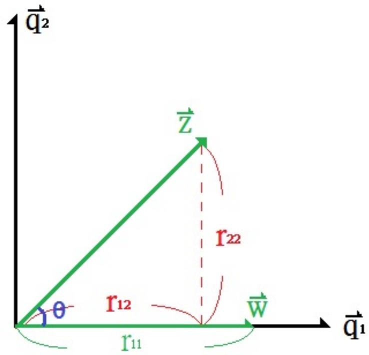
- Use new vectors $\vec{q}_1$, $\vec{q}_2$, where $\|\vec{q}_1\| = \|\vec{q}_2\| = 1$, to express $\vec{w}$, $\vec{z}$

$$\cos\theta = \frac{\vec{w}^T\vec{z}}{\|\vec{w}\|\|\vec{z}\|}$$

$$A = (\vec{w} \cdot \vec{z}) = \begin{pmatrix} \vec{q}_1 & \vec{q}_2 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \text{ by QR decomposition}$$

$$\vec{w} = r_{11}\vec{q}_1, \quad \vec{z} = r_{12}\vec{q}_1 + r_{22}\vec{q}_2, \quad \vec{q}_1 \perp \vec{q}_2, \quad \vec{q}_1 \parallel \vec{w}, \quad span(\{\vec{q}_1, \vec{q}_2\}) = span(\{\vec{w}, \vec{z}\}).$$

$$\vec{w}^T\vec{z} = (r_{11}\vec{q}_1)^T(r_{12}\vec{q}_1 + r_{22}\vec{q}_2) = r_{11} \cdot r_{12}$$



- Plane (Givens) rotation matrix:
  The formation of plane rotation matrix will be like

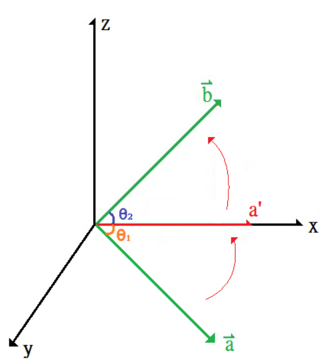
$$G_{i,j,\theta} = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

  where i is the row which contains $\cos\theta$ and $-\sin\theta$; j is the row which contains $\sin\theta$ and $\cos\theta$

  If we want to rotate $\vec{a}$ to $\vec{b}$, instead of direct rotation, we rotate to $\vec{a'}$ first, then rotate to $\vec{b}$. The rotation matrice are as following:

$$\vec{a'} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{a}$$

$$\vec{b} = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \vec{a'}$$



- Using Givens rotation to compute QR decomposition:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$G_{1(1,2,\theta_1)}A = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A^{(1)}$$

$$G_{2(1,3,\theta_2)}A^{(1)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} \end{pmatrix} = A^{(2)}$$

$$G_{3(2,3,\theta_3)}A^{(2)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} \\ 0 & a_{22}^{(3)} & a_{23}^{(3)} \\ 0 & 0 & a_{33}^{(3)} \end{pmatrix} = A^{(3)}$$

$$Q = G_1^T G_2^T G_3^T, \quad G_3 G_2 G_1 A = R, \quad A = (G_1 G_2 G_3)^T R$$

- Time complexity

$$\begin{aligned} T(m,n) &= (m-1)(4n+C) + T(m-1,n-1) \\ &= \sum_{k=1}^{n} 4k(m-1+k-n) \\ &= \sum_{k=1}^{n} 4k(m-n-1) + 4k^2 \\ &= (m-n-1) \times 4 \times \frac{n(n+1)}{2} + 4 \times \frac{n(n+1)(2n+1)}{6} \\ &\approx 2mn^2 (\text{for finding R}) \end{aligned}$$

  To find Q, the time complexity is also $2mn^2$.

- Givens rotation is uasually applied to sparse matrices.

2 Block matrix decomposition

Matrix decomposition can also use block form and parallelization to speed up the performance

- Block LU

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = LU$$

$$L = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

where L_{11}, L_{22}, L_{33} are lower triangular matrice, and U_{11}, U_{22}, U_{33} are upper triangular matrice.

$$\begin{aligned}
A_{11} &= L_{11}U_{11} & Step1 &: LU(A_{11}) \Rightarrow L_{11}, U_{11} \\
A_{21} &= L_{21}U_{11} & Step2 &: L_{21} = A_{21}U_{11}^{-1} \\
A_{31} &= L_{31}U_{11} & Step2 &: L_{31} = A_{31}U_{11}^{-1} \\
A_{12} &= L_{11}U_{12} & Step2 &: U_{12} = L_{11}^{-1}A_{12} \\
A_{22} &= L_{21}U_{12} + L_{22}U_{22} & Step3 &: LU(A_{22} - L_{21}U_{12}) \Rightarrow L_{22}, U_{22} \\
A_{32} &= L_{31}U_{12} + L_{32}U_{22} & Step4 &: L_{32} = (A_{32} - L_{31}U_{12})U_{22}^{-1} \\
A_{13} &= L_{11}U_{13} & Step2 &: U_{13} = L_{11}^{-1}A_{13} \\
A_{23} &= L_{21}U_{13} + L_{22}U_{23} & Step4 &: U_{23} = L_{22}^{-1}(A_{23} - L_{21}U_{13}) \\
A_{33} &= L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} & Step5 &: LU(A_{33} - L_{31}U_{13} - L_{32}U_{23}) \Rightarrow L_{33}, U_{33}
\end{aligned}$$

Let us have these four operations:

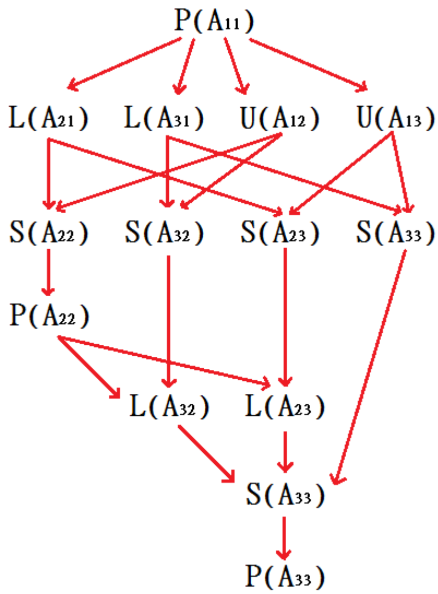
1. P(A_{ii}): LU(A_{ii}),

2. L(A_{ij}): compute(L_{ij}),

3. U(A_{ij}): compute(U_{ij}),

4. S(A_{ij}): update(A_{ij})

The following DAG (directed acyclic graph) shows that the dependence of each operation. In other words, each operation can only start to compute until all its previous operations are done.



- Block QR

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = QR, \quad Q^T Q = I$$

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{pmatrix}$$

where R_{11}, R_{22}, R_{33} are upper triangular matrices.

Using block Givens method for block matrix decomposition
(Reason: Givens method is element orient, not vector orient.)

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ 0 \end{pmatrix}$$

Block version:

$$\begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} X \\ Y \end{pmatrix} = QR$$

$$G = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & C & 0 & -S & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & S & 0 & C & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

Let us have these four operations:

1. F(A_{ij}): $A_{ij} = QR_{ij}$
2. F$\begin{pmatrix} A_{ij} \\ A_{kj} \end{pmatrix}$: $\begin{pmatrix} A_{ij} \\ A_{kj} \end{pmatrix} = QR$
3. Q(A_{ij}): apply Q to A_{ij}
4. Q$\begin{pmatrix} A_{ij} \\ A_{kj} \end{pmatrix}$: apply Q to $\begin{pmatrix} A_{ij} \\ A_{kj} \end{pmatrix}$

The following diagram shows that the dependence of each operation. In other words, each operation can only start to compute until all its previous operations are done.