Full Orthogonal Method (FOM)	Conjugate Gradient (CG)
1. $q_1 = b/\|b\|$, $Q_1 = [q_1]$, $T_1 = q_1^T A q_1$, $(Aq_1 = q_1 T_1 + \beta_1 q_2)$ 2. For $k = 1, 2, \ldots$ (a) $y_k = \|b\| T_k^{-1} e_1$. (b) If $\|r_k\| = \|\beta_k y_k(k)\|$ is small enough, Return $x_k = Q_k y_k$. (c) Expand $Q_{k+1} = [Q_k \quad q_{k+1}]$ and $T_{k+1} = \begin{bmatrix} T_k & t_k \\ \beta_k e_k^T & q_{k+1}^T A q_{k+1} \end{bmatrix}$ by Lanczos method. $(AQ_{k+1} = Q_{k+1} T_{k+1} + \beta_{k+1} q_{k+2} e_{k+1}^T)$	1. $r_0 = b - Ax_0$, $p_0 = r_0$ 2. For $k = 1, 2, \ldots$ (a) $\delta_k = \dfrac{r_{k-1}^T r_{k-1}}{r_{k-1}^T A p_{k-1}}$. (b) $x_k = x_{k-1} + \delta_k p_{k-1}$ (c) $r_k = r_{k-1} - \delta_k A p_{k-1}$ (d) If $\|r_k\|$ is small enough, Return x_k. (e) $\gamma_k = \dfrac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$ (f) $p_k = r_k + \gamma_k p_{k-1}$

Theorem 1 *CG is FOM for symmetric positive definite matrices.*

Outline of the proof (From FOM to CG)

1. $T_k = L_k U_k$
2. $y_k = T_k^{-1} \|b\| e_1 = U_k^{-1} L_k^{-1} \|b\| e_1$
3. $x_k = Q_k y_k = x_{k-1} + \delta_k p_k$
4. $r_k = r_{k-1} - A p_k$
5. $p_k = r_k + \gamma_k p_{k-1}$
6. Derive δ_k and γ_k

Step I:

$$
\begin{aligned}
T_k &= \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \beta_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = L_k U_k \\
L_k &= \begin{pmatrix} 1 & & & \\ \lambda_1 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \\
U_k &= \begin{pmatrix} \mu_1 & \beta_1 & & & \\ & \mu_2 & \beta_2 & & \\ & & \mu_3 & \beta_3 & \\ & & & \ddots & \ddots \end{pmatrix}
\end{aligned}
$$

Step II:

$$\begin{aligned}
T_k^{-1} &= U_k^{-1} L_k^{-1} \\
L_k^{-1} &= \begin{pmatrix} 1 & & & & \\ -\lambda_1 & 1 & & & \\ \lambda_1\lambda_2 & -\lambda_2 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ (-1)^{k-1}\lambda_1\lambda_2\cdots\lambda_{k-1} & \dots & \dots & -\lambda_{k-1} & 1 \end{pmatrix} \\
U_k^{-1} &= \begin{pmatrix} U_{k-1}^{-1} & -\beta_{k-1}/\mu_k U_{k-1}^{-1} e_k \\ 0 & 1/\mu_k \end{pmatrix}
\end{aligned}$$

$$y_k = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \end{pmatrix} = T_k^{-1}\|b\|e_1 = \|b\|U_k^{-1}L_k^{-1}e_1 = \|b\|U_k^{-1}z_k$$

$$z_k = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{pmatrix} = \begin{pmatrix} z_{k-1} \\ \zeta_k \end{pmatrix}$$

$$\zeta_k = -\lambda_{k-1}\zeta_{k-1} = (-1)^{k-1}\lambda_1\lambda_2\cdots\lambda_{k-1}$$

Step III:

$$\begin{aligned}
x_k &= Q_k y_k = \|b\| Q_k U_k^{-1} z_k \\
&= \begin{pmatrix} Q_{k-1} & q_k \end{pmatrix} \begin{pmatrix} U_{k-1}^{-1} & -\beta_{k-1}/\mu_k U_{k-1}^{-1} e_k \\ 0 & 1/\mu_k \end{pmatrix} \begin{pmatrix} z_{k-1} \\ \zeta_k \end{pmatrix} \\
&= \quad Q_{k-1}U_{k-1}^{-1} \quad -\beta_{k-1}Q_{k-1}U_{k-1}^{-1}e_k/\mu_k + q_k/\mu_k \quad \begin{pmatrix} z_{k-1} \\ \zeta_k \end{pmatrix} \\
&= Q_{k-1}U_{k-1}^{-1}z_{k-1} + (-\beta_{k-1}Q_{k-1}U_{k-1}^{-1}e_k + q_k)\frac{\zeta_k}{\mu_k} \\
&= x_{k-1} + (-\beta_{k-1}Q_{k-1}U_{k-1}^{-1}e_k + q_k)\frac{\zeta_k}{\mu_k}
\end{aligned}$$

Let $\hat{p}_k = (-\beta_{k-1}Q_{k-1}U_{k-1}^{-1}e_k + q_k)\frac{\zeta_k}{\mu_k}$. Observe that

$$\begin{aligned}
\hat{p}_k &= \zeta_k Q_k U_k^{-1} e_k \\
&= \frac{\beta_{k-1}\lambda_k}{\mu_k}\hat{p}_{k-1} + \frac{\zeta_k}{\mu_k} q_k
\end{aligned}$$

Step IV:

Consider the residual $r_{k+1} = b - Ax_k$.

$$\begin{aligned} r_{k+1} &= b - Ax_k \\ &= b - A(x_{k-1} + \hat{p}_k) \\ &= (b - Ax_{k-1}) - A\hat{p}_k \\ &= r_k - A\hat{p}_k \end{aligned}$$

Also,

$$\begin{aligned} r_{k+1} &= b - Ax_k \\ &= b - AQ_k y_k \\ &= b - (Q_k T_k + \beta_k q_{k+1} e_k) y_k \\ &= b - Q_k T_k y_k + \beta_k q_{k+1} \eta_k \end{aligned}$$

Since $T_k y_k = \|b\| e_1$ and $q_1 = b/\|b\|$,

$$b - Q_k T_k y_k = b - \|b\| Q_k e_1 = b - \|b\| q_1 = 0.$$

In addition, η_k is the last element of $U_k^{-1} z_k$, which is ζ_k/μ_k. As the result, $r_{k+1} = -\beta_k \zeta_k / \mu_k q_{k+1}$. From the LU decomposition, we know that $\beta_k/\mu_k = \lambda_k$. Combining with (), we have

$$r_k = \zeta_k q_k. \tag{1}$$

Since $\|q_k\| = 1$, $\|r_k\| = |\zeta_k|$.

Step V:

Let $p_k = \mu_k \hat{p}_k$.

$$\begin{aligned} p_k &= \beta_{k-1} \lambda_{k-1} \hat{p}_{k-1} + \zeta_k q_k \\ &= \frac{\beta_{k-1}}{\mu_{k-1}} \lambda_{k-1} p_{k-1} + r_k \\ &= (\lambda_{k-1})^2 p_{k-1} + r_k. \end{aligned}$$

Let $\delta_k = 1/\mu_k$ and $\gamma_k = (\lambda_{k-1})^2$.

$$\begin{cases} x_k = x_{k-1} + \delta_k p_k, \\ r_k = r_{k-1} - \delta_k A p_k, \\ p_k = \gamma_k p_{k-1} + r_k, \end{cases} \tag{2}$$

Step VI:

How to compute δ_k and γ_k?

$$\gamma_k = \lambda_{k-1}^2 = \frac{\zeta_k^2}{\zeta_{k-1}^2} = \frac{\|r_k\|}{\|r_{k-1}\|} = \frac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}$$

We know that r_k is parallel to q_k, which means they are orthogonal. Multiplying r_{k-1}^T to (2), we have

$$\delta_k = \frac{(r_{k-1}, r_{k-1})}{(r_{k-1}, Ap_k)}.$$