CS5321 Numerical Optimization Homework 8

Due June 21

1. (30%) Consider the following problem

$$\begin{aligned} \min_{\vec{x}} \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 2 \end{aligned}$$

   (a) Write the augmented Lagrangian penalty function of the problem, and its Hessian matrix (partially differentiated with respected to $\vec{x}$).

   (b) To make the augmented Lagrangian exact, i.e. having the same solution as the original problem, what should be the penalty parameter μ? (Hint: use the second order optimality condition to evaluate the Hessian matrix.)

2. (70%) Consider the following linear programming problem

$$\begin{aligned} \min_{\vec{x}} \quad & -x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 40 \\ & 2x_1 + x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   (a) Transform the problem to the standard form.

   (b) Write the dual problem of (a).

   (c) Write the KKT condition of the (b) with the relaxed complementarity slackness condition.

   (d) Write the formula of calculating the step direction $\vec{p}_k$ of the Newton's method for solving the KKT condition in (c).

   (e) Implement the Interior Point Method, as shown in Figure 1, to solve the problem with $(x_1, x_2) = (5, 30)$, and output the $\vec{x}_k, \vec{s}_k$, and $\vec{\lambda}_k$. Try different initial slack variables $\vec{s}_0$ and α_k to see the effects.

(1) Given an interior point $\vec{x}_0$ and the initial guess of slack variables $\vec{s}_0$.

(2) For $k = 0, 1, \ldots$

(3) Choose $\sigma_k \in [0, 1]$ and solve

$$\begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} \Delta x_k \\ \Delta \lambda_k \\ \Delta s_k \end{pmatrix} = \begin{pmatrix} \vec{c} - \vec{s}_k - A^T \vec{\lambda}_k \\ \vec{b} - A\vec{x}_k \\ -X^k S^k e + \sigma_k \mu_k e \end{pmatrix},$$

where $\mu_k = \vec{x}_k^T \vec{s}_k / n$, $X^k = \mathrm{diag}(\vec{x}_k)$, $S^k = \mathrm{diag}(\vec{s}_k)$.

(4) Compute α_k such that

$$(\vec{x}_{k+1}, \vec{\lambda}_{k+1}, \vec{s}_{k+1})^T = (\vec{x}_k, \vec{\lambda}_k, \vec{s}_k)^T + \alpha_k (\Delta x_k, \Delta \lambda_k, \Delta s_k)^T$$

is in the region $N(\gamma) = \{(\vec{x}, \vec{\lambda}, \vec{s}) | x_i s_i \geq \gamma \mu_k, \forall i = 1, 2, \ldots, n\}$ for some $\gamma \in (0, 1]$.

Figure 1: The interior point method for solving linear programming