CS5321 Numerical Optimization Homework 7

Due June 7

1. (40%)

   (a) Suppose G is symmetric positive definite. Prove the dual problem of a quadratic programming,

$$\begin{aligned} \min_{\vec{x}} \quad & \frac{1}{2}\vec{x}^T G\vec{x} + \vec{c}^T\vec{x} \\ \text{s.t.} \quad & A\vec{x} - \vec{b} \geq \vec{0}, \vec{x} \geq 0 \end{aligned}$$

   is

$$\begin{aligned} \max_{\vec{\lambda}} \quad & -\frac{1}{2}(A^T\vec{\lambda} - \vec{c})^T G^{-1}(A^T\vec{\lambda} - \vec{c}) + \vec{\lambda}^T\vec{b} \\ \text{s.t.} \quad & \vec{\lambda} \geq \vec{0} \end{aligned}$$

   (b) Convert the following quadratic program to the dual problem

$$\begin{aligned} \min_{x_1,x_2} \quad & q(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2.5)^2 \\ \text{s.t.} \quad & x_1 - 2x_2 + 2 \geq 0 \\ & -x_1 - 2x_2 + 6 \geq 0 \\ & -x_1 + 2x_2 + 2 \geq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

   (c) Solve the primal and the dual problem in (b) by Matlab's function `quadprog` with $\vec{x}_0 = (2, 0)^T$, and compare their results.

2. (60%) Consider the following constrained minimization problem

$$\begin{aligned} \min_{\vec{x}} \quad & e^{x_1x_2x_3x_4x_5} - \frac{1}{2}(x_1^3 + x_2^3 + 1)^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10 = 0 \\ & x_2x_3 - 5x_4x_5 = 0 \\ & x_1^3 + x_2^3 + 1 = 0 \end{aligned}$$

which has a solution $\vec{x}^* = (-1.8, 1.7, 1.9, -0.8, -0.8)^T$. Implement the SQP method, as sketched in Figure 1, in Matlab to solve the problem with $\vec{x}_0 = (-1.71, 1.59, 1.82, -0.763, -0.763)^T$.

(a) Write the Lagrangian function of the problem.

(b) Write the gradient and the Hessian of the Lagrangian function.

(c) Write the Jacobian of the constraints.

(d) At a solution (x_k, λ_k), express the problem by a quadratic program.

(e) Print out $\vec{x}_k$ of your program.

For the problem

$$\begin{aligned} \min_{\vec{x}} \quad & f(\vec{x}) \\ \text{s.t.} \quad & c(\vec{x}) = \begin{pmatrix} c_1(\vec{x}) \\ c_2(\vec{x}) \\ \vdots \\ c_m(\vec{x}) \end{pmatrix} = \vec{0} \end{aligned}$$

1. Choose an initial pair $(\vec{x}_0, \vec{\lambda}_0)$.
2. For $k = 0, 1, \ldots$ until converge

   (a) Evaluate $\nabla f(\vec{x}_k), \nabla^2_{xx}\mathcal{L}(\vec{x}_k, \vec{\lambda}_k), c(\vec{x}_k)$ and $A_k = \nabla c(\vec{x}_k)$.

   (b) Solve the local quadratic program

   $$\begin{aligned} \min_{\vec{p}} \quad & \frac{1}{2}\vec{p}^T \nabla^2_{xx}\mathcal{L}_k \vec{p} + \nabla f_k^T \vec{p} \\ \text{s.t.} \quad & A_k \vec{p} + c(\vec{x}_k) = 0 \end{aligned}$$

   to obtain $\vec{p}_k$ and $\vec{\ell}_k$.

   (c) Set $\vec{x}_{k+1} = \vec{x}_k + \vec{p}_k$ and $\vec{\lambda}_k = \vec{\ell}_k$.

Figure 1: Local Sequential Quadratic Programming (SQP) method