CS5321 Numerical Optimization Homework 5

Due May 10

1. (20%) Linear equality constraints.

   (a) Reduce the following problem to an unconstrained problem,

$$\begin{aligned} \min_{x_1,\ldots,x_6} \quad & \sin(x_3+x_6)+x_1^3+x_2^2+x_1x_3x_6+x_4x_5^2 \\ \text{subject to} \quad & 8x_1-6x_2+x_3+9x_4+4x_5=6 \\ & 3x_1+2x_2-4x_4+6x_5+4x_6=-4 \end{aligned}$$

   There is no unique solution to this problem. We can rewrite it in any form. For example, let

$$\begin{aligned} x_3 &= 6-8x_1+6x_2-9x_4-4x_5 \\ x_6 &= -1-\frac{3}{4}x_1-\frac{1}{2}x_2+x_4-\frac{3}{2}x_5 \end{aligned}$$

   The problem becomes

$$\begin{aligned} \min_{x_1,x_2,x_4,x_5} \quad & \sin(5-8\frac{3}{4}x_1+4\frac{1}{2}x_2-8x_4-4\frac{3}{2}x_5)+x_1^3+x_2^2+x_4x_5^2+ \\ & x_1(6-8x_1+6x_2-9x_4-4x_5)(-1-\frac{3}{4}x_1-\frac{1}{2}x_2+x_4-\frac{3}{2}x_5) \end{aligned}$$

   (b) Consider the general form of a constrained minimization problem with only linear equality constraints.

$$\min_{\vec{x}} f(\vec{x}) \text{ subject to } A\vec{x}=\vec{b},$$

   where $A \in \mathbb{R}^{m\times n}$ with $m<n$. Suppose A has full row rank. Prove that this problem can be reduced to an unconstrained problem with $n-m$ unknowns. (Hint: use similar technique in the simplex method for representing basic variables by nonbasic variables.)

Since A is of full row rank, we can find m linearly independent columns of A. We can gather them into a B matrix, and the rest into an N matrix, by using the permutation matrix P, $AP = [B|N]$. The elements of $\vec{x}$ can be reordered accordingly, $P^T\vec{x} = \begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix}$. Thus,

$$\vec{b} = A\vec{x} = APP^T\vec{x} = [B|N]\begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix} = B\vec{x}_B + N\vec{x}_N.$$

Since B is of full rank, it is invertible.

$$\vec{x}_B = B^{-1}(b - N\vec{x}_N).$$

The constrained problem can be rewritten as

$$\min_{\vec{x}_N} f\left(P\begin{bmatrix} B^{-1}(\vec{b} - N\vec{x}_N) \\ \vec{x}_N \end{bmatrix}\right).$$

2. (80%) Consider the following linear programming problem

$$\begin{aligned} \max_{x_1,x_2} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 4 \\ & 4x_1 + 2x_2 \le 12 \\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

   (a) Draw the figure of the constraints and use that to solve the problem.

   (b) Derive its dual problem and solve the dual problem by any means. Compare the solutions of the primal and the dual problems.

$$\begin{aligned} \min_{y_1,y_2,y_3} \quad & z = 4y_1 + 12y_2 + y_3 \\ \text{subject to} \quad & y_1 + 4y_2 - y_3 \ge 1 \\ & 2y_1 + 2y_2 + y_3 \ge 1 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

   The optimal solution is at $y_1^* = 2/6, y_2^* = 1/6, y_3^* = 0$, and $z* = 10/3$.

(c) Verify the complementarity slackness condition.

For the primal problem, we add slack variables s_1, s_2 and s_3,

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 4 \qquad (1) \\ 4x_1 + 2x_2 + s_2 &= 12 \qquad (2) \\ -x_1 + x_2 + s_3 &= 1 \qquad (3) \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

The optimal solution is at $x_1^* = 8/3, x_2^* = 2/3$, which makes $s_1^* = s_2^* = 0$, and $s_3^* = 3$. It can be verified easily that $y_i^* s_i^* = 0$ for $i = 1, 2, 3$.

For the dual problem, we add slack variables t_1, t_2,

$$\begin{aligned} y_1 + 4y_2 - y_3 + t_1 &= 1 \qquad (1) \\ 2y_1 + 2y_2 + y_3 + t_2 &= 1 \qquad (2) \\ y_1, y_2, y_3, t_1, t_2 &\geq 0 \end{aligned}$$

At the optimal solution, $t_1^* = t_2^* = 0$, by which $x_i^* t_i^* = 0$ for $i = 1, 2$.

(d) Transform the problem to the standard form.

$$\begin{aligned} \min_{x_1, x_2} \quad & z = -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 4 \\ & 4x_1 + 2x_2 + x_4 = 12 \\ & -x_1 + x_2 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

(e) Solve it by the simplex method, as provided in Figure 1, using $\vec{x}_0 = (0, 0)$. Indicate $B_k, N_k, \vec{s}_k, \vec{d}_k, p_k, q_k$ and γ_k in each step.

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}, \vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 12 \\ 1 \end{pmatrix}$$

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~~~~~~~~~~~~~~~~~~~~ k=0 ~~~~~~~~~~~~~~~~~~~~
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$\mathcal{B}_0 = \{3,4,5\}, \mathcal{N}_0 = \{1,2\}, \vec{c}_0 = (-1,-1,0,0,0)^T$,

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, N_0 = \begin{pmatrix} 1 & 2 \\ 4 & 2 \\ -1 & 1 \end{pmatrix}.$$

$\vec{c}_N = (-1,-1)^T, \vec{c}_B = (0,0,0)^T, \vec{x}_N = (0,0)^T, \vec{x}_B = (4,12,1)^T$.

$$\vec{s}_0 = \vec{c}_N - N_0^T B_0^{-1} \vec{c}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Select $q_0 = 1$. (You can also choose $q_0 = 2$ since both coefficients are -1.)

$$\vec{d}_0 = B_0^{-1} A(:,1) = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

Since only $\vec{d}_0(1) > 0$ and $\vec{d}_0(2) > 0$. Compute $\vec{x}_B(1)/\vec{d}_0(1) = 4/1 = 4$ and $\vec{x}_B(2)/\vec{d}_0(2) = 12/4 = 3$. Therefore $\gamma_0 = 3$ and $i_p = 2$.

$$\vec{x}_1([3,4,5,1,2]) = \begin{pmatrix} 4 \\ 12 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ -4 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \\ 3 \\ 0 \end{pmatrix}.$$

~~~~~~~~~~~~~~~~~~~~ k=1 ~~~~~~~~~~~~~~~~~~~~

$\mathcal{B}_1 = \{3,1,5\}, \mathcal{N}_1 = \{4,2\}, \vec{c}_N = (0,-1)^T, \vec{c}_B = (0,-1,0)^T$.

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 1 \end{pmatrix}, N_1 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

$\vec{x}_N = \vec{x}_1([4,2]) = (0,0)^T, \vec{x}_B = \vec{x}_1([3,1,5]) = (1,3,4)^T$.

$$\vec{s}_1 = \vec{c}_N - N_1^T B_1^{-1} \vec{c}_B = \begin{pmatrix} 1/4 \\ -3/4 \end{pmatrix}.$$

Select $q_1 = 2$.

$$\vec{d}_1 = B_1^{-1} A(:,2) = \begin{pmatrix} 1.5 \\ .5 \\ 1.5 \end{pmatrix}$$
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Since only $\vec{d}_1(1) > 0$ and $\vec{d}_1(2) > 0$. Compute $\vec{x}_B(1)/\vec{d}_1(1) = 1/1.5 = 2/3$ and $\vec{x}_B(2)/\vec{d}_1(2) = 3/.5 = 6$. Therefore $\gamma_1 = 2/3$ and $i_p = 1$.

$$\vec{x}_2([3,1,5,4,2]) = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix} + 2/3 \begin{pmatrix} -1.5 \\ -.5 \\ -1.5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 8/3 \\ 3 \\ 0 \\ 2/3 \end{pmatrix}.$$

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~~~~~~~~~~~~~~~~~~~~~ k=2 ~~~~~~~~~~~~~~~~~~~~~
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$\mathcal{B}_2 = \{2,1,5\}, \mathcal{N}_2 = \{4,3\}, \vec{c}_N = (0,0)^T, \vec{c}_B = (-1,-1,0)^T$.

$$B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & -1 & 1 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\vec{x}_N = \vec{x}_2([4,3]) = (0,0)^T, \vec{x}_B = \vec{x}_2([2,1,5]) = (2/3, 8/3, 3)^T$.

$$\vec{s}_2 = \vec{c}_N - N_2^T B_2^{-1} \vec{c}_B = \begin{pmatrix} .5 \\ .5 \end{pmatrix}.$$

Since both elements of $\vec{s}_2$ are nonnegative, we found the optimal solution.

(f) Use Matlab function `libprog` to solve the problem. The default method used by `libprog` is the interior point method. How to change it to the simplex method?

(1) Given a basic feasible point $\vec{x}_0$ and the corresponding index set $\mathcal{B}_0$ and $\mathcal{N}_0$.

(2) For $k = 0, 1, \ldots$

(3) Let $B_k = A(:, \mathcal{B}_k), N_k = A(:, \mathcal{N}_k)$, $\vec{x}_B = \vec{x}_k(\mathcal{B}_k), \vec{x}_N = \vec{x}_k(\mathcal{N}_k)$, and $\vec{c}_B = \vec{c}_k(\mathcal{B}_k), \vec{c}_N = \vec{c}_k(\mathcal{N}_k)$.

(4) Compute $\vec{s}_k = \vec{c}_N - N_k^T B_k^{-1} \vec{c}_B$ (pricing)

(5) If $\vec{s}_k \geq 0$, return the solution $\vec{x}_k$. (found optimal solution)

(6) Select $q_k \in \mathcal{N}_k$ such that $\vec{s}_k(i_q) < 0$, where i_q is the index of q_k in $\mathcal{N}_k$

(7) Compute $\vec{d}_k = B_k^{-1} A_k(:, q_k)$. (search direction)

(8) If $\vec{d}_k \leq 0$, return `unbounded`. (unbounded case)

(9) Compute $[\gamma_k, i_p] = \min\limits_{i, \vec{d}_k(i) > 0} \dfrac{\vec{x}_B(i)}{\vec{d}_k(i)}$ (ratio test)

(The first return value is the minimum ratio;
the second return value is the index of the minimum ratio.)

(10) $$x_{k+1} \begin{pmatrix} \mathcal{B} \\ \mathcal{N} \end{pmatrix} = \begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix} + \gamma_k \begin{pmatrix} -\vec{d}_k \\ \vec{e}_{i_q} \end{pmatrix}$$

($\vec{e}_{i_q} = (0, \ldots, 1, \ldots, 0)^T$ is a unit vector with i_qth element 1.)

(11) Let the i_pth element in $\mathcal{B}$ be p_k. (pivoting)
$\mathcal{B}_{k+1} = (\mathcal{B}_k - \{p_k\}) \cup \{q_k\}$, $\mathcal{N}_{k+1} = (\mathcal{N}_k - \{q_k\}) \cup \{p_k\}$

Figure 1: The simplex method for solving (minimization) linear programming