CS5321 Numerical Optimization Homework 2

Due March 25

1. (15%) Prove that $\vec{a}^T\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos(\theta)$ for $\vec{a}, \vec{b} \in \mathbb{R}^n$ and θ is the angle between $\vec{a}$ and $\vec{b}$. (Hint: Let $\vec{c} = \vec{a} - \vec{b}$, and use the relation of $\|\vec{a}\|$, $\|\vec{b}\|$, $\|\vec{c}\|$ in a triangle to derive the result.)

2. (15%) Consider a function $f(x,y) = \begin{cases} \frac{xy}{x+y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. Show that its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$, but the directional derivative $D(f,[1,1]) \neq \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$ at $(0,0)$.

3. (20%) Compute the LDL decomposition of the matrix
$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 2 & 3 & 4 \end{pmatrix}.$$
(Hint: Use pivoting to stablize the computation, and put the pivotings into a permutation matrix P, such that $PAP^T = LDL^T$.)

4. (50%) Let $f(x,y) = \frac{1}{2}x^2 + \frac{9}{2}y^2$. This is a positive definite quadratic with minimizer at $(x^*, y^*) = (0,0)$.
   (a) Derive the gradient g and the Hessian H of f.
   (b) Write Matlab codes to implement the steepest descent method and Newton's method with $\vec{x}_0 = (9,1)$, and compare their convergent results. The formula of the steepest descent method is
   $$\vec{x}_{k+1} = \vec{x}_k - \frac{\vec{g}_k^T\vec{g}_k}{\vec{g}_k^T H_k \vec{g}_k}\vec{g}_k,$$
   and the formula of Newton's method is
   $$\vec{x}_{k+1} = \vec{x}_k - H_k^{-1}\vec{g}_k,$$
   where $\vec{g}_k = g(\vec{x}_k)$ and $H_k = H(\vec{x}_k)$.
   (c) Draw the trace of $\{\vec{x}_k\}$ for the steepest descent method and Newton's method. Figure 1 gives an example code for trace drawing.

```
function draw_trace()

% draw the contour of the function z = (x*x+9*y*y)/2;
step = 0.1;
X = 0:step:9;
Y = -1:step:1;

n = size(X,2);
m = size(Y,2);

Z = zeros(m,n);
for i = 1:n
    for j = 1:m
        Z(j,i) = f(X(i),Y(j));
    end
end

contour(X,Y,Z,100)

% plot the trace
%   You can record the trace of your results and use the following
%   code to plot the trace.
xk = [9 8 8 7 7 6 6 5 5 4 4 3 3 2 2];
yk = [.5 .5 -.5 -.5 .5 .5 -.5 -.5 .5 .5 -.5 -.5 .5 .5 -.5];
hold on;   % this is important!! This will overlap your plots.
plot(xk,yk,'-','LineWidth',3);
hold off;

% function definition
    function z = f(x,y)
        z = (x*x+9*y*y)/2;
    end
end
```

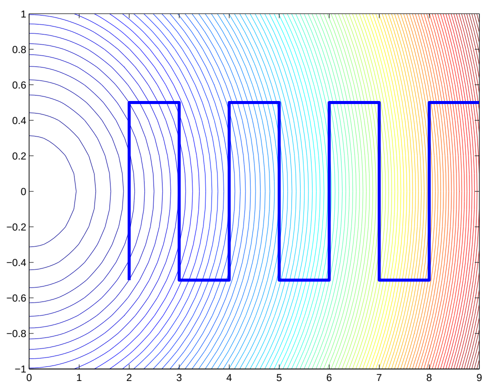

Figure 1: Function contour and a trace of (xk,yk).