Derivation of the local quadratic model

$$\min f(x) \text{ s.t. } c(x) = 0$$

The Lagrangian of above problem is

$$L(x, \lambda) = f(x) - \lambda c(x)$$

The KKT condition imposes the following equations

$$\nabla L_x(x, \lambda) = \nabla f(x) - \lambda \nabla c(x) = 0$$

$$\nabla L_\lambda(x, \lambda) = c(x) = 0$$

Let $F(x,\lambda) = \begin{pmatrix} \nabla_x L \\ \nabla_\lambda L \end{pmatrix} = \begin{pmatrix} \nabla f - \lambda \nabla c \\ c \end{pmatrix}$

F = 0 can be solved by the Newton's method

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \lambda_k \end{pmatrix} + \begin{pmatrix} p_k \\ \mu_k \end{pmatrix}$$

$$\nabla F(x_k, \lambda_k) \begin{pmatrix} p_k \\ \mu_k \end{pmatrix} = -F(x_k, \lambda_k)$$

where $\nabla F_k \equiv \nabla F(x_k, \lambda_k) = \begin{pmatrix} \nabla^2_{xx} L_k & -\nabla c_k \\ \nabla c_k^T & 0 \end{pmatrix}$

The above equation can be rewritten as

$$\begin{pmatrix} \nabla^2_{xx} L_k & \nabla c_k \\ \nabla c_k^T & 0 \end{pmatrix} \begin{pmatrix} -p_k \\ \mu_k \end{pmatrix} = \begin{pmatrix} \nabla f_k - \lambda_k \nabla c_k \\ c_k \end{pmatrix}$$

The equation of the first row block is

$$-\nabla^2_{xx}L_x p_k + \nabla c_k \mu_k = \nabla f_k - \lambda_k \nabla c_k$$

$$-\nabla^2_{xx}L_x p_k + \nabla c_k (\lambda_k + \mu_k) = \nabla f_k$$

$$-\nabla^2_{xx}L_x p_k + \nabla c_k \lambda_{k+1} = \nabla f_k$$

The entire system can be rewritten as

$$\begin{pmatrix} \nabla^2_{xx}L_k & \nabla c_k \\ \nabla c_k^T & 0 \end{pmatrix} \begin{pmatrix} -p_k \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} \nabla f_k \\ c_k \end{pmatrix}$$

If it is a KKT system of a quadratic programming, the original problem is (chap 16)

$$\min_p \qquad \frac{1}{2}p^T \nabla^2 L(x_k, \lambda_k) p + p^T \nabla f(x_k)$$

$$\text{s.t.} \qquad \nabla c(x_k)^T p + c(x_k) = 0$$

Example from N&S book

$$\min f(x) = e^{3x_1+4x_2} \text{ s.t. } c(x) = x_1^2 + x_2^2 - 1 = 0$$

$$\nabla f = \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{3x_1+4x_2} \qquad \nabla c = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$\nabla^2 f = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} e^{3x_1+4x_2} \qquad \nabla^2 c = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$L(x,\lambda) = f(x) - \lambda c(x)$$

$$\nabla L_x(x,\lambda) = \begin{pmatrix} 3f(x) - 2\lambda x_1 \\ 4f(x) - 2\lambda x_2 \end{pmatrix}$$

$$\nabla L_{xx}(x,\lambda) = \begin{pmatrix} 9f(x) - 2\lambda & 12f(x) \\ 12f(x) & 16f(x) - 2\lambda \end{pmatrix}$$

Initial guess $x_0 = (-0.7, -0.7)^T, \lambda_0 = -0.01$

$$f(x_0) = 0.0074, c(x_0) = -0.02$$

$$\nabla f(x_0) = \begin{pmatrix} .022 \\ .029 \end{pmatrix} \quad \nabla c(x_0) = \begin{pmatrix} -1.4 \\ -1.4 \end{pmatrix}$$

$$\nabla^2 L(x_0, \lambda_0) = \begin{pmatrix} .087 & .089 \\ .089 & .139 \end{pmatrix}$$

$$p = \begin{pmatrix} .14 \\ -.15 \end{pmatrix}, x_1 = x_0 + p = \begin{pmatrix} -.55 \\ -.85 \end{pmatrix}$$

$$\lambda_1 = -0.14808$$

$$x^* = \begin{pmatrix} -.6 \\ -.8 \end{pmatrix}, \lambda^* = -2.5e^{-5} \approx -.168$$

Quadratic merit function

$$M(x) = f(x) + 10c(x)^T c(x)$$

$$x_0 = (-0.7, -0.7)^T, f(x_0) = .0074, c(x_0) = -.02$$

$$M(x_0) = f(x_0) + 10c(x_0)^T c(x_0) = .011447$$

$$x_1 = (-0.55, -0.85)^T, f(x_1) = .0061, c(x_1) = .044$$

$$M(x_1) = f(x_1) + 10c(x_1)^T c(x_1) = .025961$$

Set step length α=2/3, and let $x_1 = x_0 + \alpha p$

$$\hat{x}_1 = (-0.61, -0.80)^T, f(\hat{x}_1) = .0063, c(\hat{x}_1) = .013$$

$$M(\hat{x}_1) = f(\hat{x}_1) + 10c(\hat{x}_1)^T c(\hat{x}_1) = .00826$$