Examples from textbook

$$\min_x \frac{1}{2} x^T \begin{bmatrix} 6 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} x + \begin{bmatrix} -8 & -3 & -3 \end{bmatrix} x$$

$$\text{s.t.} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Solution is at $x^* = (2, -1, 1)^T, \lambda^* = (3, -2)^T$

KKT matrix $\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 & 1 & 0 \\ 2 & 5 & 2 & 0 & 1 \\ 1 & 2 & 4 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

Schur-complement method

$$\begin{aligned} -Gp + A^T\lambda^* &= g \\ (AG^{-1}A^T)\lambda^* &= AG^{-1}g - h \end{aligned}$$

$$AG^{-1}A^T = \begin{bmatrix} 0.4819 & 0.1084 \\ 0.1084 & 0.3494 \end{bmatrix}$$

Let x=(0,0,0)^T

$$h = Ax - b = -b = \begin{bmatrix} -3 & 0 \end{bmatrix}^T$$

$$g = c + Gx = c = \begin{bmatrix} -8 & -3 & -3 \end{bmatrix}^T$$

$$\lambda^* = (AG^{-1}A^T)^{-1}(AG^{-1}g - h) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$p = G^{-1}(A^T\lambda - g) = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T = x^* - x = x^*$$

Null space method

$$A^T = QR = \begin{bmatrix} -0.71 & 0.41 & -0.58 \\ 0 & -0.82 & -0.58 \\ -0.71 & -0.41 & 0.58 \end{bmatrix} \begin{bmatrix} -1.41 & -0.71 \\ & -1.22 \\ & \end{bmatrix}$$

$$\begin{aligned} R^T Q_1^T p &= R^T p_1 = -h \\ (Q_2^T G Q_2) p_2 &= -Q_2^T G Q_1 p_1 - Q_2^T g \\ R\lambda^* &= Q_1^T (g + Gp) \end{aligned}$$

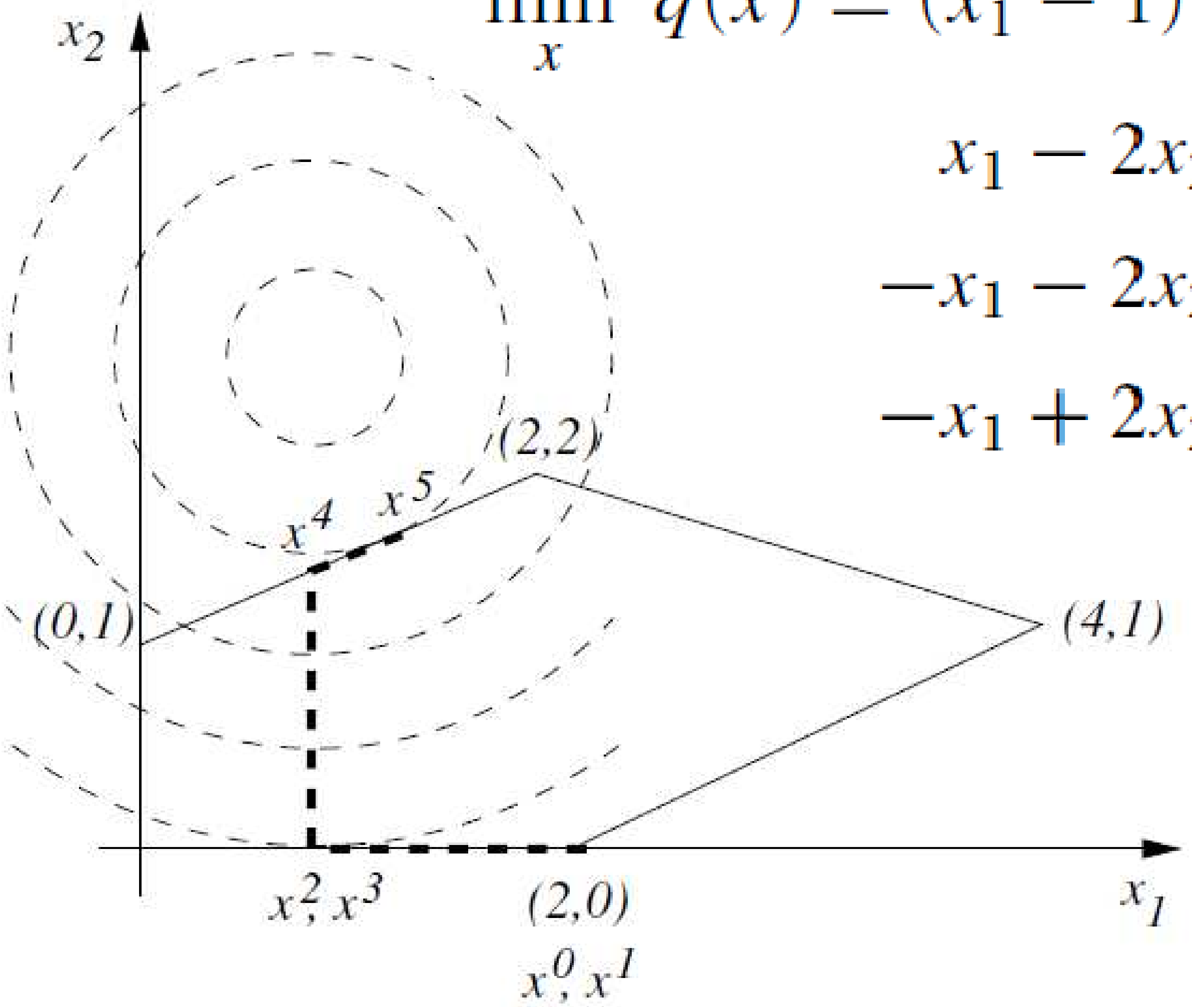
$$p_1 = -R^{-T} h = \begin{bmatrix} 2.12 & -1.22 \end{bmatrix}^T$$

$$p_2 = -(Q_2^T G Q_2)^{-T} (Q_2^T G Q_1 p_1 - Q_2^T g) = 0$$

$$p = Q_1 p_1 + Q_2 p_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$$

$$\lambda^* = R^{-1} Q_1^T (g + Gp) = \begin{bmatrix} 3 & -2 \end{bmatrix}^T$$

Active set example



$$\min_x \; q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 \geq 0, \quad (1)$$

$$-x_1 - 2x_2 + 6 \geq 0, \quad (2)$$

$$-x_1 + 2x_2 + 2 \geq 0, \quad (3)$$

$$x_1 \geq 0, \quad (4)$$

$$x_2 \geq 0. \quad (5)$$

Initial guess is $x^0 = (2,0), \mathbf{W}_0 = \{3,5\}$

Solve $$\min_x \quad (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 \geq 0$$

$$-x_1 - 2x_2 + 6 \geq 0$$

$$-x_1 + 2x_2 + 2 = 0$$

$$x_1 \geq 0, x_2 = 0$$

Solution is p=0, since x_1,x_2 are fixed by {3,5}.

Check Lagrangian multipliers of 3 and 5.

If both are positive, the solution is found (KKT condition).

Otherwise, remove one constraint.

How to find Lagrangian multipliers?

- We have seen this in the algorithms for equality constraints: solving KKT system.

$$\begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

- Since $g=c+Gx$ and $p=x^*-x$, $A^T\lambda^* = Gx^* + c$

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}, x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\lambda^* = \begin{bmatrix} \lambda_3 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Not an optimal solution.

Remove the 3rd constraint.

Next one $x^1 = (2,0), \mathbf{W}_1 = \{5\}$

Solve $$\min_x \quad (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 \geq 0$$

$$-x_1 - 2x_2 + 6 \geq 0$$

$$-x_1 + 2x_2 + 2 \geq 0$$

$$x_1 \geq 0, x_2 = 0$$

Solution is p=(−1,0).

Check if x^1+p=(1,0) violates

any inequality constraints.

Next one $x^2 = (1,0), \mathbf{W}_2 = \{5\}$

Solve

$$\min_x \quad (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 \geq 0$$

$$-x_1 - 2x_2 + 6 \geq 0$$

$$-x_1 + 2x_2 + 2 \geq 0$$

$$x_1 \geq 0, x_2 = 0$$

Solution is *p*=0, Check optimality

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_5 = \begin{bmatrix} 0 \\ -5 \end{bmatrix}, \lambda_5 = -5$$

(1,0) is not an optimal solution. Remove the 5th constraint.

Next one $x^3 = (1,0), \mathbf{W}_3 = \{\}$

Solve $$\min_x \quad (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 \geq 0$$
$$-x_1 - 2x_2 + 6 \geq 0$$
$$-x_1 + 2x_2 + 2 \geq 0$$
$$x_1 \geq 0, x_2 \geq 0$$

Solution is *p*=(0,2.5). Check if x^3+p=(1, 2.5) violates any inequality constraints.

$$x_1 - 2x_2 + 2 = 1 - 5 + 2 = -2 < 0$$

Set step length α=0.6, so $x^3 + \alpha p$=(1,1.5).

Add 1 to the working set.

Next one $x^4 = (1, 1.5), \mathbf{W}_4 = \{1\}$

Solve

$$\min_x \quad (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 = 0$$

$$-x_1 - 2x_2 + 6 \geq 0$$

$$-x_1 + 2x_2 + 2 \geq 0$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution is p=(0.4,0.2). Check if x^4+p=(1.4, 1.7) violates any inequality constraints.

Next one $x^5 = (1.4, 1.7), \mathbf{W}_5 = \{1\}$

Solve

$$\min_x \quad (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$x_1 - 2x_2 + 2 = 0$$

$$-x_1 - 2x_2 + 6 \geq 0$$

$$-x_1 + 2x_2 + 2 \geq 0$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution is *p*=0. Check optimality.

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \lambda_1 = \begin{bmatrix} 0.8 \\ -1.6 \end{bmatrix}, \lambda_1 = 0.8$$

We found the solution.

Interior point method

$$\min_x \; q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

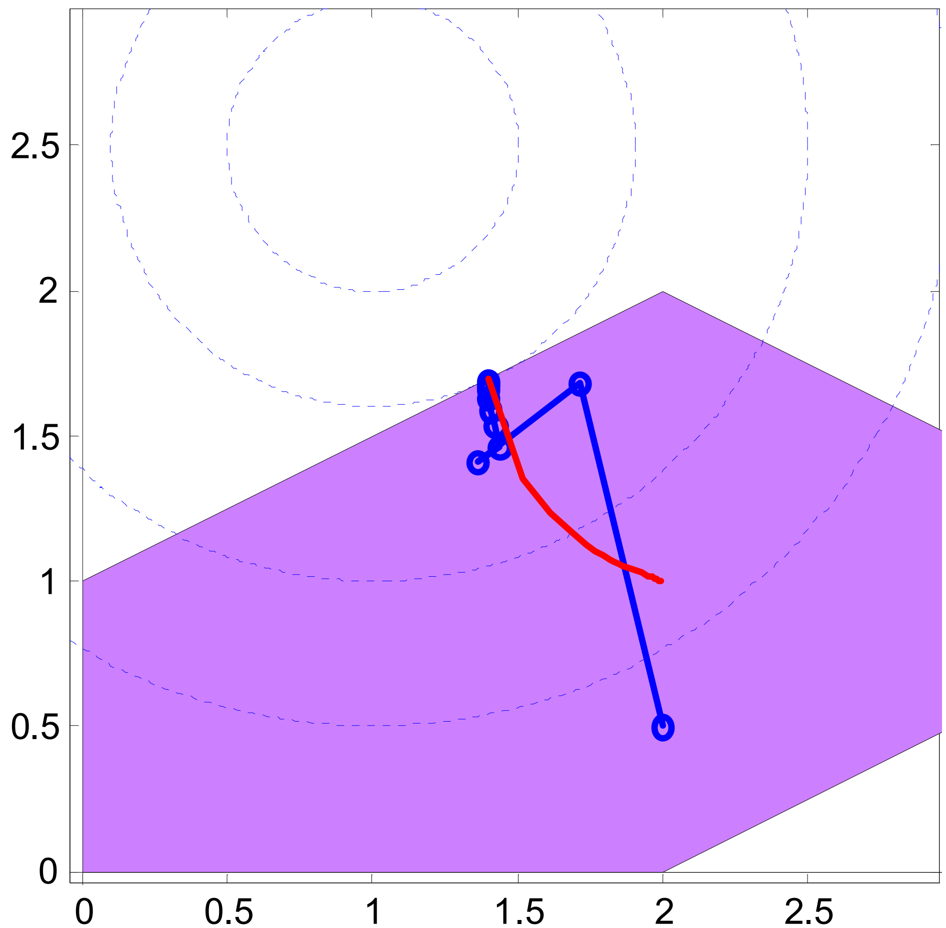
$$x_1 - 2x_2 + 2 \geq 0,$$
$$-x_1 - 2x_2 + 6 \geq 0,$$
$$-x_1 + 2x_2 + 2 \geq 0,$$
$$x_1 \geq 0,$$
$$x_2 \geq 0.$$

$$\min_x \quad \frac{1}{2}x^T G x + c^T x + d$$
$$s.t. \quad Ax \geq b$$

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ -1 & -2 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -6 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} G & 0 & -A^T \\ A & -I & 0 \\ 0 & \Lambda & Y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \lambda \end{pmatrix} = \begin{pmatrix} -r_d \\ -r_p \\ -\Lambda Y e + \sigma \mu e \end{pmatrix}$$



Gradient projection method

$$\min_{x_1,x_2,x_3} f=(x_1+2x_2-1)^2+(x_2+2x_3-1)^2+(x_3-3/4)^2$$

$$0 \le x_1, x_2, x_3 \le 1$$

$$\min_x f(x) = \frac{1}{2} x^T G x + c^T x + d$$

$$G = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & 4 \\ 0 & 4 & 10 \end{pmatrix}, c = \begin{pmatrix} -2 \\ -6 \\ -11/2 \end{pmatrix}, d = \frac{23}{16}$$

Gradient: $g = Gx + c = \begin{pmatrix} 2x_1 + 4x_2 - 2 \\ 4x_1 + 10x_2 + 4x_3 - 6 \\ 4x_2 + 10x_3 - 11/2 \end{pmatrix}$

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, g = \begin{pmatrix} -2 \\ -6 \\ -11/2 \end{pmatrix}$$

$$t_i = \begin{cases} (x_i - u_i)/g_i & \text{if } g_i < 0 \\ (x_i - l_i)/g_i & \text{if } g_i > 0 \\ \infty & \text{otherwise} \end{cases}$$

$$t_1 = \frac{1}{2}, t_2 = \frac{1}{6}, t_3 = \frac{2}{11}$$

$$x_i(t) = \begin{cases} x_i - tg_i & \text{if } t \leq t_i \\ x_i - t_i g_i & \text{otherwise} \end{cases}$$

Since $t_2 < t_3 < t_1$, for $t \in (0, t_2), (t_2, t_3), (t_2, t_3)$

$$x(t) = \begin{pmatrix} 2t \\ 6t \\ 11/2t \end{pmatrix}, \begin{pmatrix} 2t \\ 1 \\ 11/2t \end{pmatrix}, \begin{pmatrix} 2t \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1/3 \\ 1 \\ 11/12 \end{pmatrix} \rightarrow \begin{pmatrix} 4/11 \\ 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- For $0 \le t \le 1/6$, minimizer is at $x_1^* = (0.13, 0.41, 0.37)^T$, where $f(x_1^*) = .0682$
  - $t = 0.0682$
- For $1/6 \le t \le 2/11$, $f(x_2^*) = 5.167$
  - $t = 1/6$
- For $2/11 \le t \le 1/2$, $f(x_3^*) = 5.922$
  - $t = 2/11$