Linear constraints

$$\min \ \sin(x_1 + x_2) + x_3^2 + \frac{1}{3}(x_4 + x_5^4 + x_6/2)$$

$$\text{subject to} \quad \begin{aligned} 8x_1 - 6x_2 + x_3 + 9x_4 + 4x_5 &= 6 \\ 3x_1 + 2x_2 - x_4 + 6x_5 + 4x_6 &= -4. \end{aligned}$$

- Let $x=(x_3,x_6|x_1,x_2,x_4,x_5)^T$, and solve x_3,x_6

$$\begin{bmatrix} x_3 \\ x_6 \end{bmatrix} = -\begin{bmatrix} 8 & -6 & 9 & 4 \\ \frac{3}{4} & \frac{1}{2} & \frac{-1}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 6 \\ -1 \end{bmatrix}.$$

$$\min_{x_1,x_2,x_4,x_5} \ \sin(x_1 + x_2) + (8x_1 - 6x_2 + 9x_4 + 4x_5 - 6)^2$$

$$+ \frac{1}{3}(x_4 + x_5^4 - [(1/2) + (3/8)x_1 + (1/4)x_2 - (1/8)x_4 + (3/4)x_5])$$

General solution to Ax=b

$$A = \begin{bmatrix} 1 & 0 & 8 & -6 & 9 & 4 \\ 0 & 4 & 3 & 2 & -1 & 6 \end{bmatrix} = [B\ N]$$

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} + \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix} x_N$$

$$Y = \begin{bmatrix} -B^{-1}b \\ 0 \end{bmatrix}$$

is a solution to Ax=b

$$AY = -BB^{-1}b + 0 = b$$

$$Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}$$

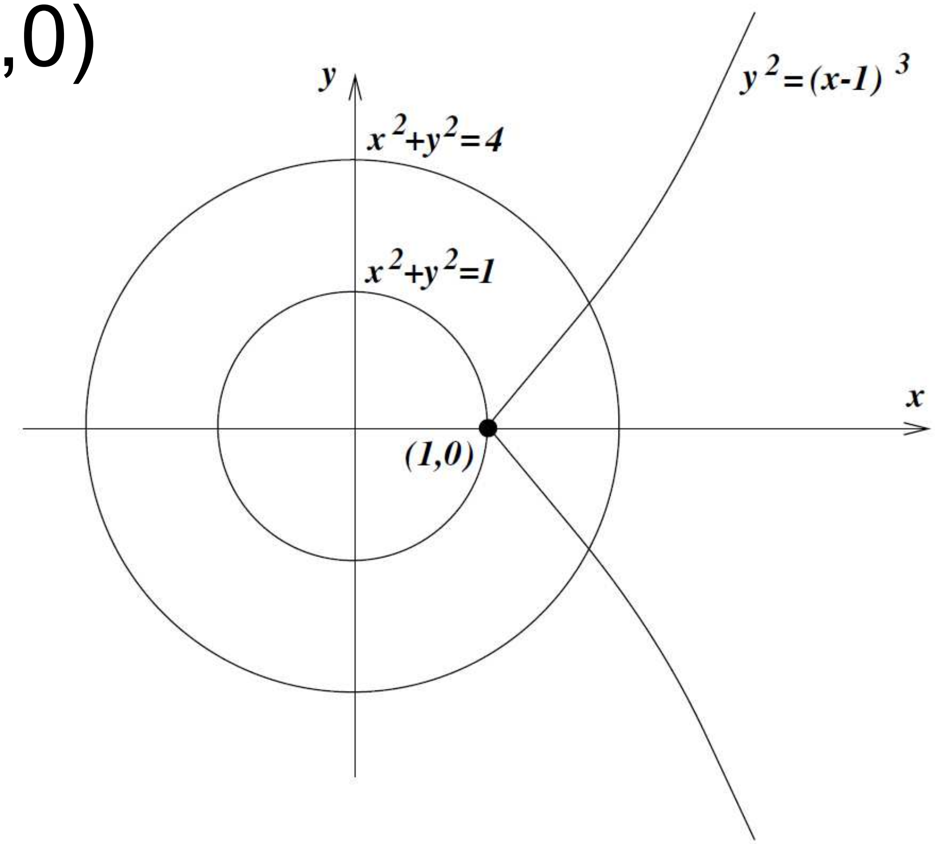
is a null space of A^T

$$AZ = -BB^{-1}N + N = 0$$

Nonlinear elimination

$$\min x^2 + y^2 \qquad \text{subject to } (x-1)^3 = y^2.$$

- The solution is at (1,0)
- But replacing y^2 by $(x-1)^3$ results



$$h(x) = x^2 + (x-1)^3.$$

  - Solution is at $-\infty$

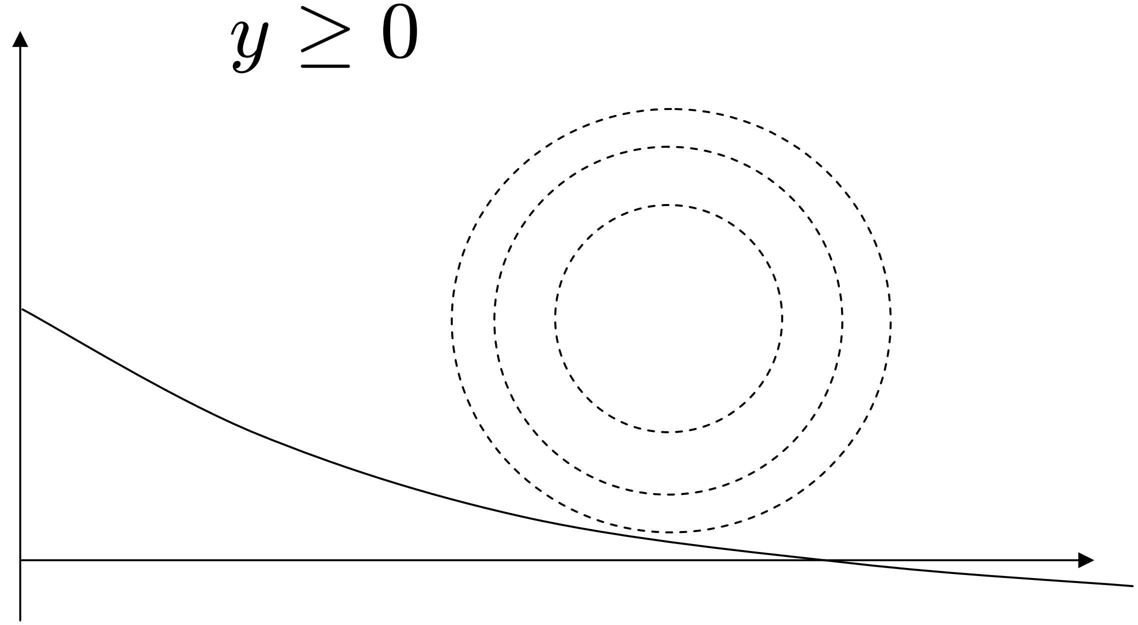
Difficulty of inequality constraints

$$\min_{x,y} f(x,y) = (x-2)^2/2 + (y-.5)/2$$

$$\text{s.t.} \qquad (x+1)^{-1} - y - 0.25 \geq 0$$

$$x \geq 0$$

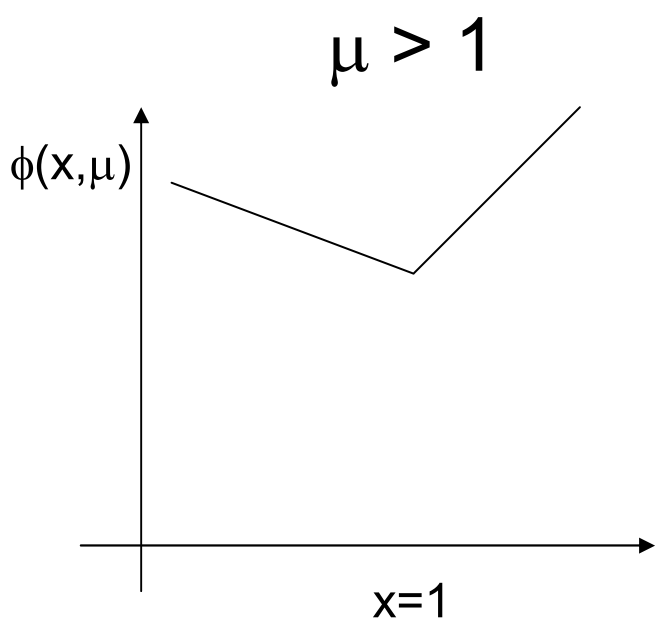
$$y \geq 0$$

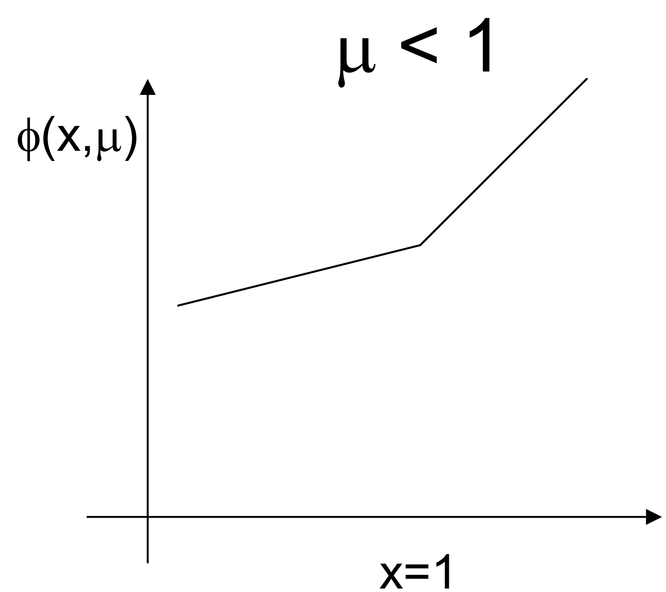


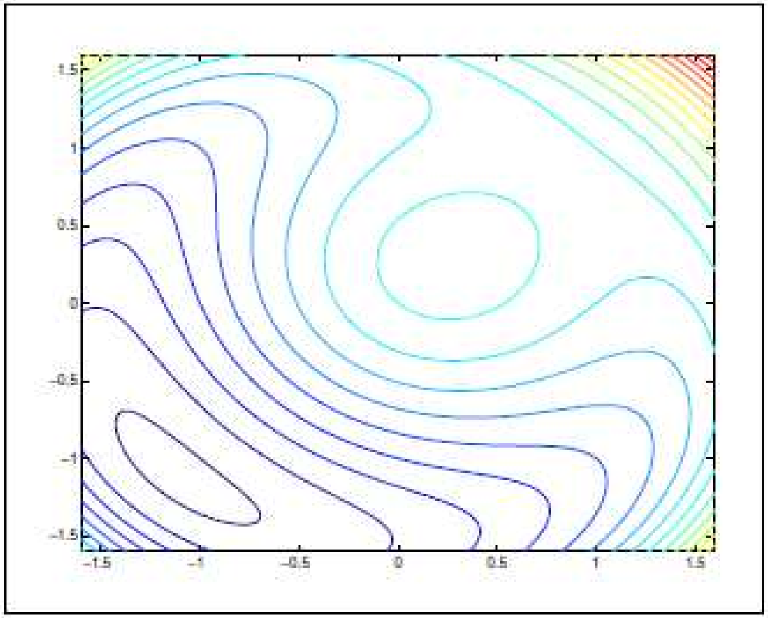
Merit function

$$\min x \text{ subject to } x \geq 1$$

$$\phi(x,\mu) = x + \mu[x-1]^- = \begin{cases} (1-\mu)x + \mu & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$





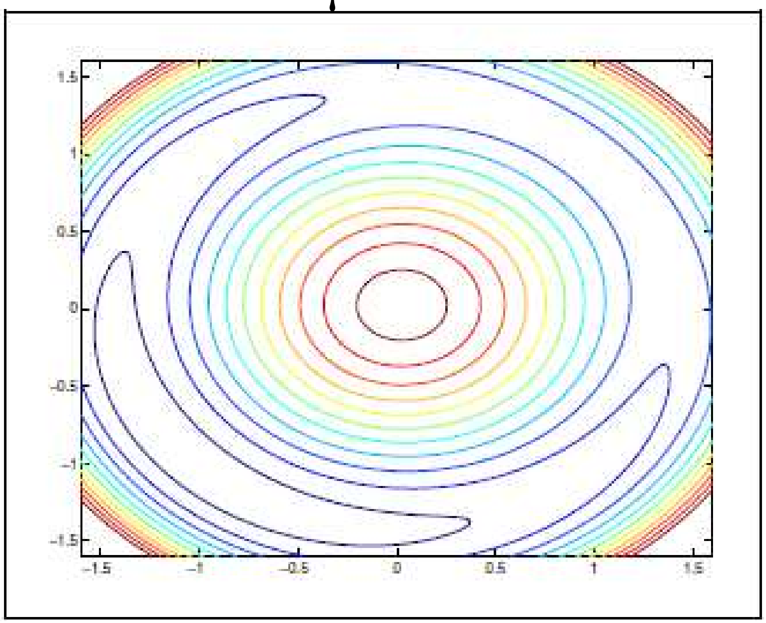


μ=1

$$\min\ x_1 + x_2$$

$$x_1^2 + x_2^2 - 2 = 0,$$

the solution is $(-1, -1)^T$

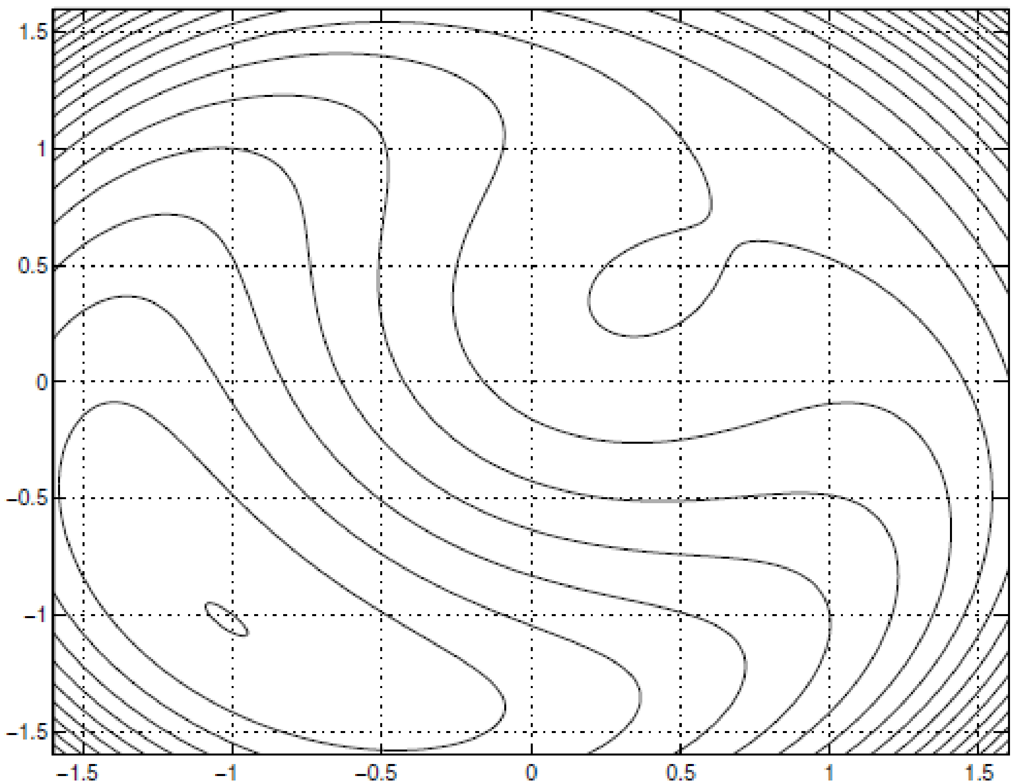


μ=10

$$Q(x;\mu) = x_1 + x_2 + \frac{\mu}{2}\left(x_1^2 + x_2^2 - 2\right)^2$$

Augmented Lagrangian

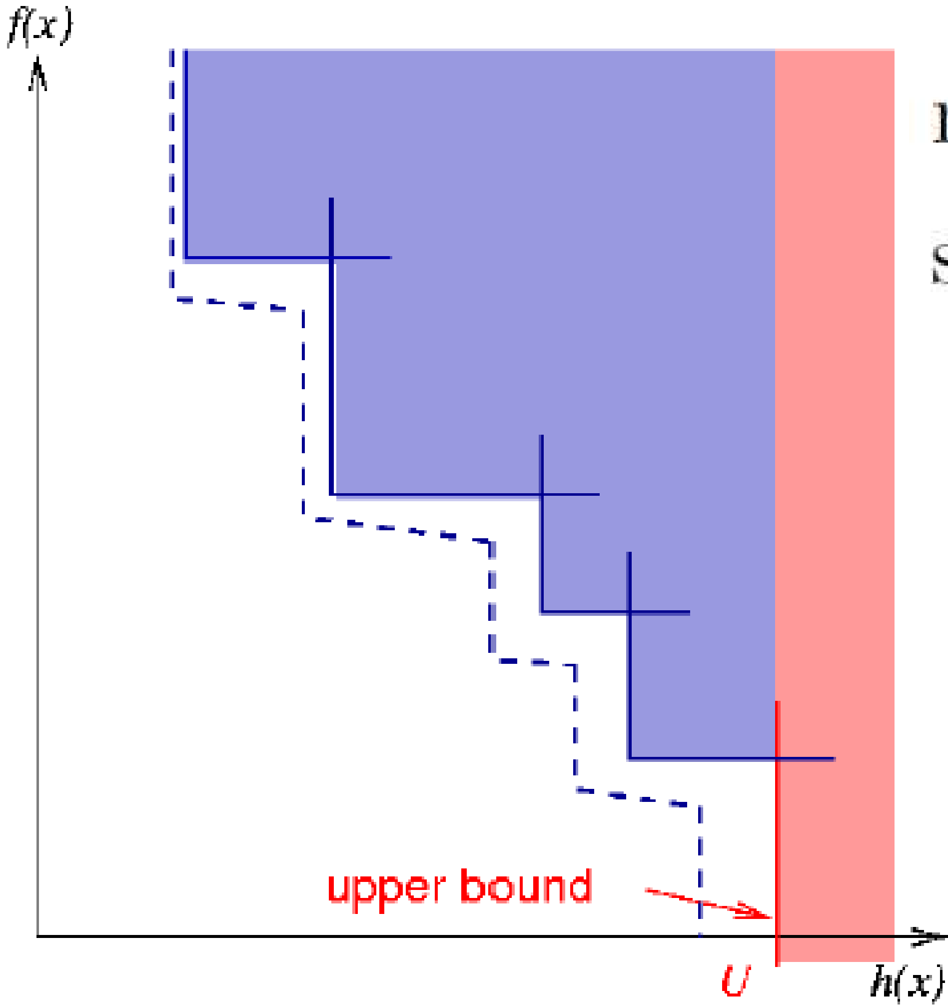
$$\mathcal{L}_A(x, \lambda; \mu) = x_1 + x_2 - \lambda(x_1^2 + x_2^2 - 2) + \frac{1}{2}\mu(x_1^2 + x_2^2 - 2)^2$$



$\lambda = -0.4$ and $\mu = 1$

$x_k \approx (-1.02, -1.02)$

Filters



$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c(x) \geq 0, \end{aligned}$$

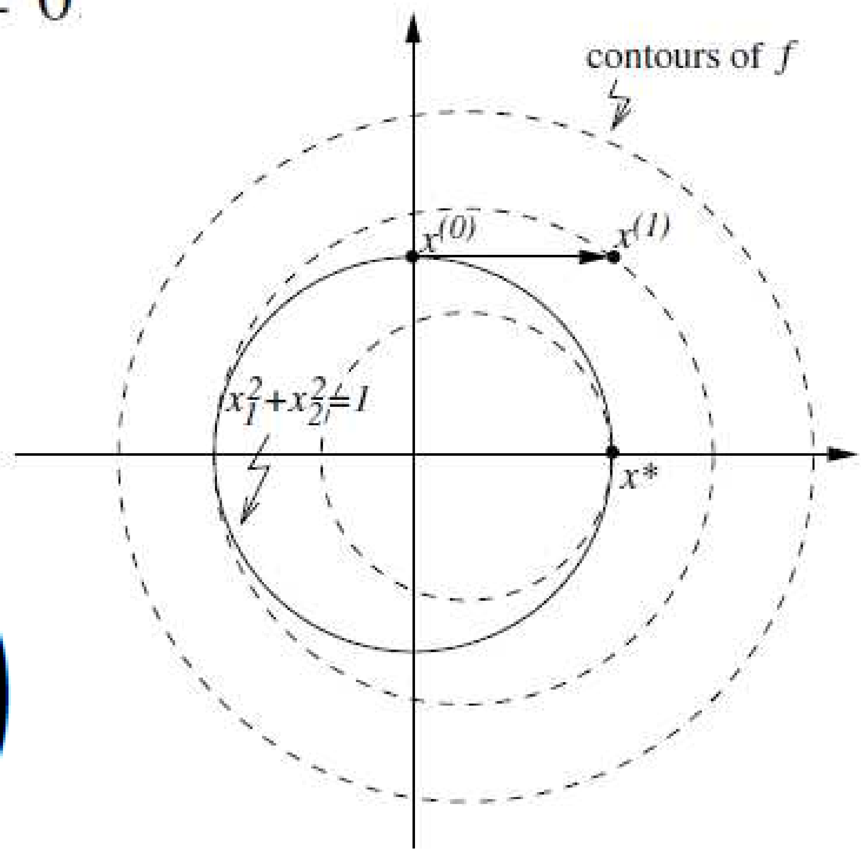
$$h(x) = \sum_{i \in \mathbf{I}} [c_i(x)]^-$$

Maratos effect

$$\min \; f(x_1, x_2) = 2(x_1^2 + x_2^2 - 1) - x_1$$

$$\text{subject to} \quad x_1^2 + x_2^2 - 1 = 0$$

$$x^* = (1, 0)$$



$$x_k = (\cos\theta, \sin\theta)^T$$

$$f(x_k) = -\cos\theta$$

$$\nabla f(x_k) = \begin{pmatrix} 4\cos\theta - 1 \\ 4\sin\theta \end{pmatrix}$$

- Suppose the search direction is

$$p_k = \begin{pmatrix} \sin^2\theta \\ -\sin\theta\cos\theta \end{pmatrix} \quad x_k + p_k = \begin{pmatrix} \cos\theta + \sin^2\theta \\ \sin\theta(1-\cos\theta) \end{pmatrix}$$

$$\|x_k + p_k - x^*\|_2 = 2\sin^2(\theta/2) \quad \frac{\|x_k + p_k - x^*\|_2}{\|x_k - x^*\|_2^2} = \frac{1}{2}$$

$$\|x_k - x^*\|_2 = 2|\sin(\theta/2)|$$

- But

$$f(x_k + p_k) = \sin^2\theta - \cos\theta > -\cos\theta = f(x_k),$$

$$c(x_k + p_k) = \sin^2\theta > c(x_k) = 0,$$

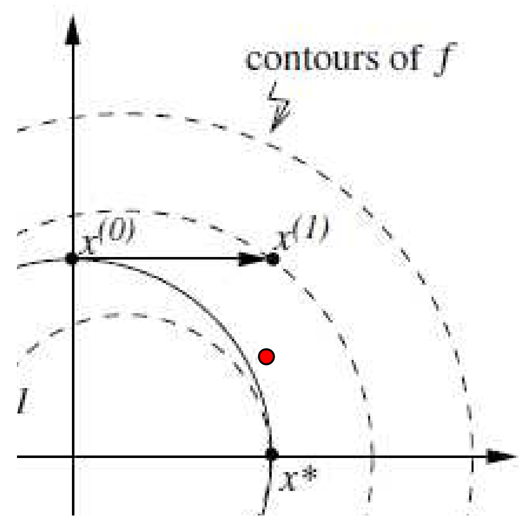
Second order correction

- The Jacobian of $c(x_k)$ is $A_k = [2x_1, 2x_2]$
- At $x_k=(0,1)^T$, $p_k = (1,0)^T$, $A_k = [0, 2]$

$$p^* = -A_k^T(A_kA_k^T)^{-1}c(x_k + p_k) = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

– $x_k + p_k + p_k^* = (1,1/2)$

$$\begin{aligned} f(x_k + p_k + p_k^*) &= -1/2 \\ c(x_k + p_k + p_k^*) &= 1/2 \end{aligned}$$



Non-monotone techniques

- Minimize the merit function at x=(1,1),μ=10

$$\phi(x) = 2(x_1^2 + x_2^2 - 1) - x_1 + 10|x_1^2 + x_2^2 - 1|$$

$$\nabla\phi(x) = \begin{bmatrix} 23 \\ 24 \end{bmatrix}$$

- Compute α to minimize $\phi(x - \alpha\nabla\phi)$
  - α = 0.042534
  - $x - \alpha\nabla\phi = (0.787, 0.744)^T$.