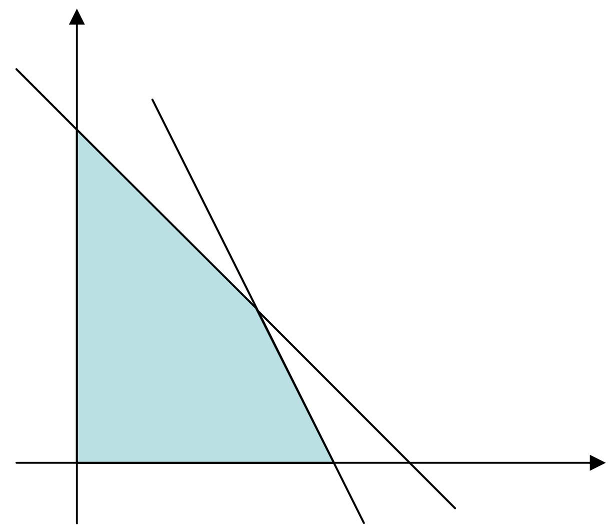
Example from Prof. Lin's note

$$\min_{x} \quad -x_1 + x_2$$

$$\text{s.t.} \quad x_1 + x_2 \leq 40$$

$$2x_1 + x_2 \leq 60$$

$$x_1, x_2 \geq 0$$



Add slack variables

$$\min_{x} \quad -x_1 + x_2 + 0x_3 + 0x_4$$

$$\text{s.t.} \quad \begin{cases} x_1 + x_2 + x_3 & = 40 \\ 2x_1 + x_2 \quad + x_4 & = 60 \\ x_1, x_2, x_3, x_4 & \geq 0. \end{cases}$$

Standard form

$$\min_x \quad c^T x$$

$$\text{s.t.} \quad A^T x = b$$

$$x \geq 0$$

$$c = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} b = \begin{pmatrix} 40 \\ 60 \end{pmatrix}$$

Starting from $\mathcal{B}$={2,4}, $\mathcal{N}$={1,3}

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad c = \begin{pmatrix} c_B \\ c_N \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4 \\ x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \\ 0 \\ 0 \end{pmatrix}$$

$$z = c^T x = 40$$

Pricing

- The constraints are

$$Ax = Bx_B + Nx_N = b$$

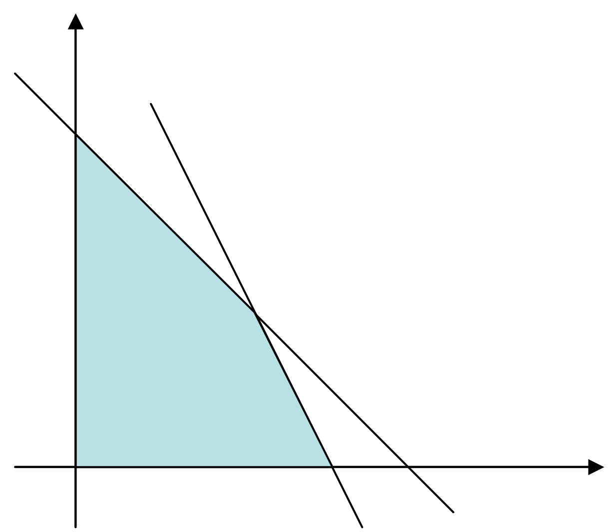
$$x_B = B^{-1}b - B^{-1}Nx_N$$

- The objective is

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N$$

$$= 40 + (-2 \ \ -1) x_N = 40 - 2x_1 - x_3$$

- One can increase x_1 or x_3 to reduce z
  - select x_1 (q=1)

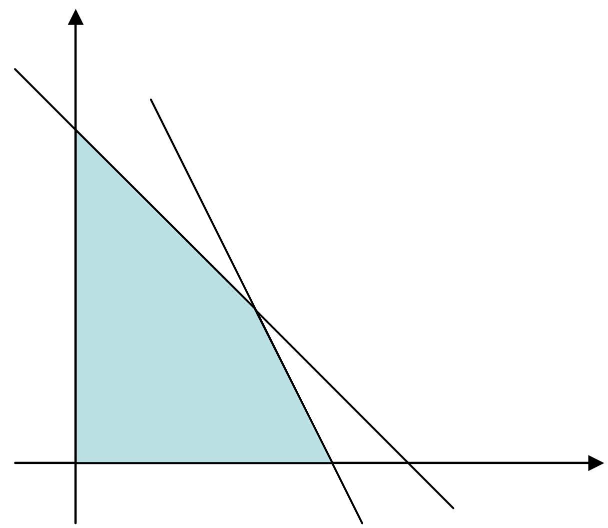


Ratio test

- How much x_1 can increase?
  - Represent x_B as a function of x_1.
  - (from constraint) $x_B = B^{-1}b - B^{-1}Nx_N$

$$x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = B^{-1}b - B^{-1}N \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 40 - x_1 \\ 20 - x_1 \end{pmatrix}$$

  - x_4 becomes zero first
  - $x_1 = 20$, and $p=2$.



Pivoting

- $\mathcal{B}$ ={2,4}, $\mathcal{N}$ ={1,3} $\Rightarrow$ $\mathcal{B}^+$ ={2,1}, $\mathcal{N}^+$ ={4,3}

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow B^+ = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = B + \begin{pmatrix} 1 \\ 1 \end{pmatrix} (0\ 1)$$

$$N = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \Rightarrow N^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \Rightarrow (B^+)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

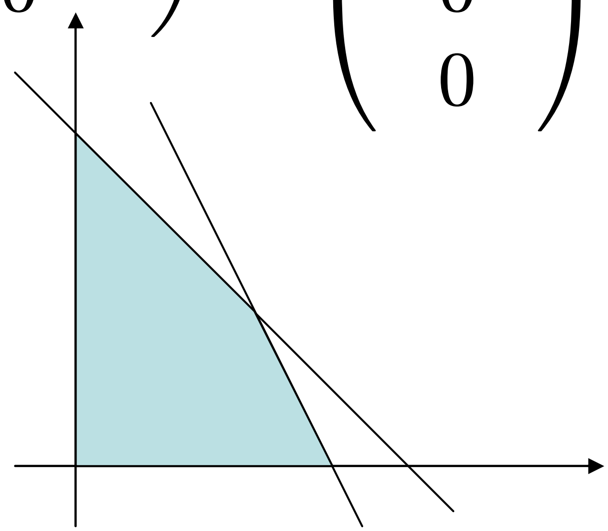
$$(B^+)^{-1} = B^{-1} - \frac{(B^{-1}N(:,1) - e_2)e_2^T B^{-1}}{1 + e_2^T(B^{-1}N(:,1) - e_2)}$$

Second point: $\mathcal{B}$ ={2,1}, $\mathcal{N}$ ={4,3}

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} c_B \\ c_N \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix}$$

$$z = c^T x = 0$$



Pricing

- The constraints are

$$Ax = Bx_B + Nx_N = b$$

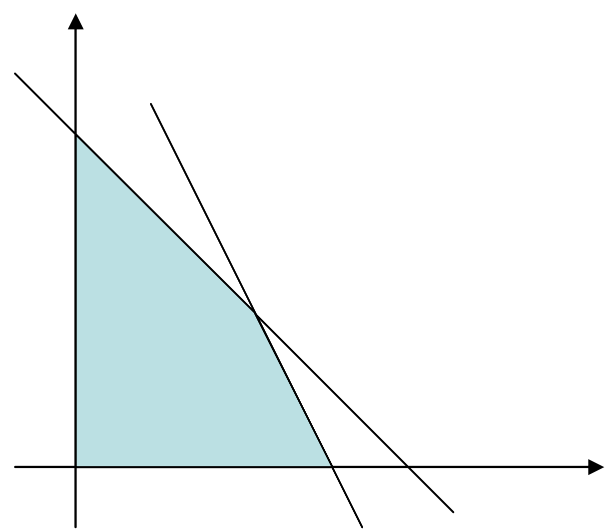
$$x_B = B^{-1}b - B^{-1}Nx_N$$

- The objective is

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N$$

$$= 0 + (0 \;\; -3) x_N = 0 - 0x_4 - 3x_3$$

- One can increase x_3 to reduce z
  – select x_3 (q=2)

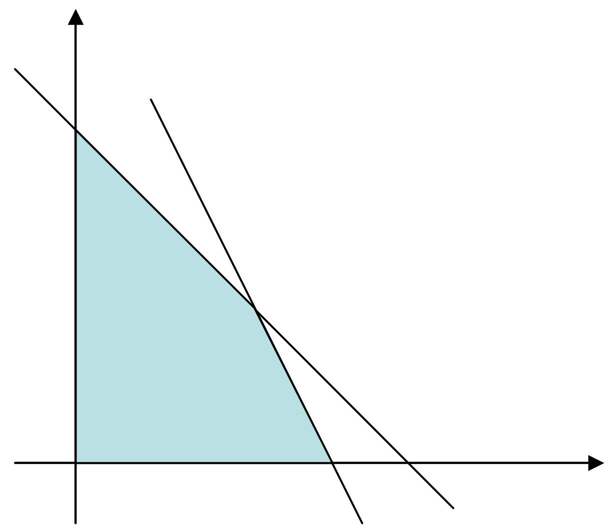


Ratio test

- How much x3 can increase?
  - Represent x_B as a function of x_3.
  - (from constraint) $x_B = B^{-1}b - B^{-1}Nx_N$

$$x_B = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = B^{-1}b - B^{-1}N \begin{pmatrix} 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 - 2x_3 \\ 20 + x_3 \end{pmatrix}$$

  - x_2 becomes zero first
  - x_3 = 10, and p=1.



Pivoting

- $\mathcal{B}$ ={2,1}, $\mathcal{N}$ ={4,3} $\Rightarrow$ $\mathcal{B}^+$ ={3,1}, $\mathcal{N}^+$ ={4,2}

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow B^+ = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = B + \begin{pmatrix} 0 \\ -1 \end{pmatrix} (1\ 0)$$

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow N^+ = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow (B^+)^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}$$

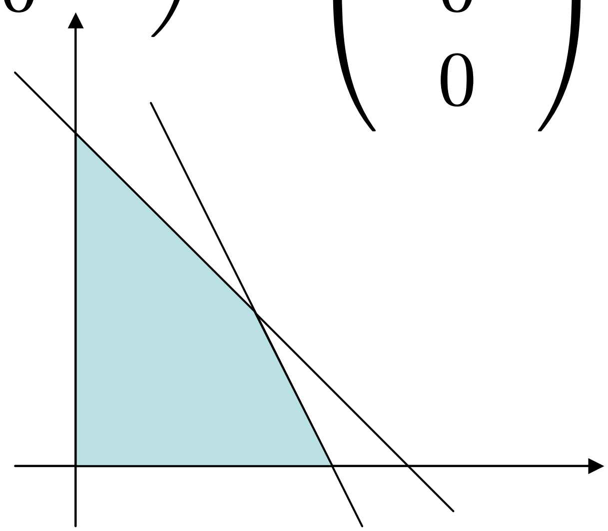
$$(B^+)^{-1} = B^{-1} - \frac{(B^{-1}N(:,2) - e_1)e_1^T B^{-1}}{1 + e_1^T(B^{-1}N(:,2) - e_1)}$$

Third point: $\mathcal{B}$={3,1}, $\mathcal{N}$={4,2}

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, c = \begin{pmatrix} c_B \\ c_N \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \\ 0 \\ 0 \end{pmatrix}$$

$$z = c^T x = -30$$



Pricing

- The constraints are

$$Ax = Bx_B + Nx_N = b$$
$$x_B = B^{-1}b - B^{-1}Nx_N$$

- The objective is

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N$$
$$= -30 + \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \end{pmatrix} x_N = -30 + x_2/2 + 3x_4/2$$

- One CANNOT increase x_2 or x_4 to reduce z
  – We found the optimal solution