# CS3331 Numerical Methods

Quiz 6, Dec 12

Name: ______________________, ID: ______________________

Decompose $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$, where $\mathbf{L}$ is the lower triangular part; $\mathbf{D}$ is the diagonal part; and $\mathbf{U}$ is the upper triangular part. The formulas of three iterative methods are

- Jacobi method: $\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}(-\mathbf{L} - \mathbf{U})\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b}$
- Gauss-Seidel method: $\mathbf{x}^{(k+1)} = -(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}\mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b}$
- Successive over relaxation (SOR): $\mathbf{x}^{(k+1)} = (\mathbf{D}+\omega\mathbf{L})^{-1}\left[((1-\omega)\mathbf{D} - \omega\mathbf{U})\mathbf{x}^{(k)} + \omega\mathbf{b}\right]$

1. True or false. (20pt)

   a ( F ) The diagonal elements of $\mathbf{D}^{-1}(-\mathbf{L} - \mathbf{U})$ are the reciprocals of $\mathbf{A}$'s diagonal elements.

   b ( F ) The eigenvalues of a strictly diagonal dominant matrix are all positive.

   c ( T ) The eigenvalues of a symmetric positive definite matrix are all positive.

   d ( F ) If $\|\mathbf{D}^{-1}(-\mathbf{L} - \mathbf{U})\| > 1$, then Jacobi method cannot converge.

   e ( T ) If $\mathbf{A}$ is symmetric and strictly diagonal dominant, then Gauss-Seidel method guarantees convergence.

   f ( F ) Gauss-Seidel method always converges faster than Jacobi method.

   g ( T ) The determinant of $(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$ equals to 0.

   h ( F ) In SOR, $0 < \omega < 2$ can guarantee the convergence.

   i ( T ) The determinant of $(\mathbf{D} + \omega\mathbf{L})^{-1}((1-\omega)\mathbf{D} - \omega\mathbf{U})$ is $(1-\omega)^n$, where $n$ is the size of $\mathbf{A}$.

   j ( F ) The spectral radius of $\mathbf{A}$ is the largest singular value of $\mathbf{A}$.

2. The new vector of SOR $\mathbf{x}^{(k+1)}$ is a linear combination of the previous one $\mathbf{x}^{(k)}$ and the vector generated by Gauss-Seidel.

$$\mathbf{x}^{(k+1)} = (1-\omega)\mathbf{x}^{(k)} + \omega[-\mathbf{D}^{-1}\mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b}].$$

Prove that $\mathbf{x}^{(k+1)} = (\mathbf{D}+\omega\mathbf{L})^{-1}\left[((1-\omega)\mathbf{D} - \omega\mathbf{U})\mathbf{x}^{(k)} + \omega\mathbf{b}\right]$. (10pt)
(*No partial credits. Giving up or leaving it blank get 3 points.*)

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= (1-\omega)\mathbf{x}^{(k)} + \omega[-\mathbf{D}^{-1}\mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b}] \\
\mathbf{D}^{-1}(\mathbf{D}+\omega\mathbf{L})\mathbf{x}^{(k+1)} &= [(1-\omega)\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{U}]\mathbf{x}^{(k)} + \omega\mathbf{D}^{-1}\mathbf{b} \\
(\mathbf{D}+\omega\mathbf{L})\mathbf{x}^{(k+1)} &= [(1-\omega)\mathbf{D} - \omega\mathbf{U}]\mathbf{x}^{(k)} + \omega\mathbf{b} \\
\mathbf{x}^{(k+1)} &= (\mathbf{D}+\omega\mathbf{L})^{-1}\left[((1-\omega)\mathbf{D} - \omega\mathbf{U})\mathbf{x}^{(k)} + \omega\mathbf{b}\right]
\end{aligned}$$

3. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{x}^{(0)} = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 4 \end{pmatrix}$. Use Gauss-Seidel method to compute $\mathbf{x}^{(1)}$. (20pt)

$$\begin{aligned}
x_1^{(1)} &= -1/2 * 0 - 1/2 * 4 + 0/2 = -2 \\
x_2^{(1)} &= 0/2 * (-2) - 1/2 * 4 + 2/2 = -1 \\
x_3^{(1)} &= -1/2 * (-2) + 0/2 = 1 \\
x_4^{(1)} &= -1/2 * (-2) - 1/2 * 1 + 2/2 = 1.5
\end{aligned}$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 1.5 \end{pmatrix}$$