CS3331 Numerical Methods

Quiz 4, Nov 14th

Name: __________________________, ID: __________________________

1. Prove a Householder matrix $\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^T$ is orthogonal (10pt).
   (*No partial credit will be given to any incomplete proof. If you leave it blank or write "Give up", you can get* **3 point**.)

$$\begin{aligned} \mathbf{H}^T\mathbf{H} &= (\mathbf{I} - 2\mathbf{v}\mathbf{v}^T)(\mathbf{I} - 2\mathbf{v}\mathbf{v}^T) \\ &= \mathbf{I} - 2\mathbf{v}\mathbf{v}^T - 2\mathbf{v}\mathbf{v}^T + 4\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T \\ &= \mathbf{I} - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}(\mathbf{v}^T\mathbf{v})\mathbf{v}^T \\ &= \mathbf{I} - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}\mathbf{v}^T = \mathbf{I} \end{aligned}$$

   $\mathbf{H}^T = \mathbf{H}^{-1}$, so $\mathbf{H}$ is orthogonal.

2. What is the orthogonal projection matrix $\mathbf{A}$ that projects vectors to the **null space** of $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. (20pt)

   Let $\mathbf{v} = \mathbf{x}/\|\mathbf{x}\| = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$.

   The matrix projects vectors to the null space of $\mathbf{x}$ is

$$\mathbf{A} = \mathbf{I} - \mathbf{v}\mathbf{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4/9 & 2/9 & 4/9 \\ 2/9 & 1/9 & 2/9 \\ 4/9 & 2/9 & 4/9 \end{pmatrix} = \begin{pmatrix} 5/9 & -2/9 & -4/9 \\ -2/9 & 8/9 & -2/9 \\ -4/9 & -2/9 & 5/9 \end{pmatrix}$$

3. Compute the QR decomposition of $\mathbf{A} = \begin{pmatrix} 4 & 9 & 1 & 3 \\ 0 & 8 & 0 & 5 \\ 0 & 0 & 7 & 2 \\ 0 & 6 & 0 & 10 \end{pmatrix}$. Make sure that all the diagonal elements of the R-factor are nonnegative. (20pt)

   You can use any methods to compute the QR decomposition, as long as the answer is correct. The simplest way is using Givens rotation.

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & -0.6 \\ 0 & 0 & 1 & 0 \\ 0 & +0.6 & 0 & 0.8 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 4 & 9 & 1 & 3 \\ 0 & 10 & 0 & 10 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$