CS3331 Numerical Methods

Quiz 10, Jan 13

Name: ________________________, ID: ________________________

1. Suppose $f(1) = 1, f(2) = 5, f(3) = -3$.

   (a) Use central difference to estimate $f'(2)$ (10pt)

   $$f'(2) \approx \frac{f(3) - f(1)}{3 - 1} = -2$$

   (b) Use three point method to estimate $f''(2)$ (10pt)

   $$f''(2) \approx \frac{f(3) - 2f(2) + f(1)}{(2-1)^2} = -12$$

   (c) Use composite trapezoid rule (2 panels) to estimate $\int_1^3 f(x)dx$ (10pt)

   $$\int_1^3 f(x)dx \approx \frac{1}{2}[f(3) + 2f(2) + f(1)] = 4$$

   (d) Use Simpson's rule to estimate $\int_1^3 f(x)dx$ (10pt)

   $$\int_1^3 f(x)dx \approx \frac{3-1}{6}[f(3) + 4f(2) + f(1)] = 6$$

2. Find w_1, w_2, w_3 and x_1, $0 < x_1 < 1$, such that

$$w_1 f(0) + w_2 f(x_1) + w_3 f(1)$$

computes exactly $\int_0^1 f(x)dx$ for polynomial $f(x)$ of degree ≤ 3. (10pt)

$$\begin{aligned}
\int_0^1 1dx &= x|_0^1 &= 1 &= w_1 + w_2 + w_3 \quad (1)\\
\int_0^1 xdx &= \frac{1}{2}x^2|_0^1 &= 1/2 &= x_1 w_2 + w_3 \quad (2)\\
\int_0^1 x^2dx &= \frac{1}{3}x^3|_0^1 &= 1/3 &= x_1^2 w_2 + w_3 \quad (3)\\
\int_0^1 x^3dx &= \frac{1}{4}x^4|_0^1 &= 1/4 &= x_1^3 w_2 + w_3 \quad (4)
\end{aligned}$$

(2)−(3) $\Rightarrow x_1 w_2 - x_1^2 w_2 = 1/6$
(3)−(4) $\Rightarrow x_1^2 w_2 - x_1^3 w_2 = 1/12$

$$x_1 w_2 - x_1^2 w_2 = 2(x_1^2 w_2 - x_1^3 w_2)$$

Since $x_1 \neq 0$, $w_2(2x_1^2 - 3x_1 + 1) = 0$. $x_1 = 1/2$
(Another solution $x_1 = 1$ does not satisfy the requirement.)
Substitute x_1 back to get $w_2 = 2/3$, $w_3 = 1/6$, and $w_1 = 1/6$.

You can see this exactly equals to the Simpson's rule,

$$\frac{1}{6}(f(0) + 4f(.5) + f(1)),$$

which explains why the Simpson's rule can have $O(h^5)$ error.