CS 3331 Numerical Methods

Lecture 9: Fourier Methods

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Outline

- Trigonometric functions
- Fourier interpolation and approximation
- Fast Fourier transform (FFT)

Trigonometric Functions

Some useful identities

$$\begin{aligned}
\sin(a+2m\pi) &= \sin(a)\\
\cos(a+2m\pi) &= \cos(a)\\
\sin(a+(2m+1)\pi) &= -\sin(a)\\
\cos(a+(2m+1)\pi) &= -\cos(a)\\
\sin(a+b) &= \sin(a)\cos(b)+\sin(b)\cos(a)\\
\sin(a-b) &= \sin(a)\cos(b)-\sin(b)\cos(a)\\
\sin(a)\cos(b) &= \frac{1}{2}(\sin(a+b)+\sin(a-b))\\
\cos(a+b) &= \cos(a)\cos(b)-\sin(a)\sin(b)\\
\cos(a-b) &= \cos(a)\cos(b)+\sin(a)\sin(b)\\
\cos(a)\cos(b) &= \frac{1}{2}(\cos(a+b)+\cos(a-b))\\
\sin(a)\sin(b) &= \frac{1}{2}(\cos(a-b)-\cos(a+b))
\end{aligned}$$

Euler's formula

$$e^{ix} = \cos x + i \sin x, \text{ where } i = \sqrt{-1}$$

- Proved by Taylor's series (at 0)

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{//odd function } f(-x) = -f(x)$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \text{//even function } f(-x) = f(x)$$

$$\begin{aligned} e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \cos x + i \sin x \end{aligned}$$

Orthogonal functions

$\{1, \cos(x), \ldots, \cos((n-1)x), \sin(x), \ldots, \sin((n-1)x)\}$ are orthogonal on $[-\pi, \pi]$

Assume $a \neq b$

$$
\begin{aligned}
\langle 1, \cos ax \rangle &= \int_{-\pi}^{\pi} \cos ax dx = \frac{1}{a} \sin ax|_{-\pi}^{\pi} = 0 \\
\langle 1, \sin ax \rangle &= \int_{-\pi}^{\pi} \sin ax dx = -\frac{1}{a} \cos ax|_{-\pi}^{\pi} = 0 \\
\langle \sin ax, \cos bx \rangle &= \int_{-\pi}^{\pi} \sin ax \cos bx dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(a+b)x + \sin(a-b)x dx = 0 \\
\langle \cos ax, \cos bx \rangle &= \int_{-\pi}^{\pi} \cos ax \cos bx dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(a+b)x + \cos(a-b)x dx = 0 \\
\langle \sin ax, \sin bx \rangle &= \int_{-\pi}^{\pi} \sin ax \sin bx dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(a-b)x - \cos(a+b)x dx = 0
\end{aligned}
$$

$$\langle 1, 1\rangle = \int_{-\pi}^{\pi} 1 dx = x|_{-\pi}^{\pi} = 2\pi$$

For $a = b$

$$\begin{aligned}
\langle \sin ax, \cos bx\rangle &= \int_{-\pi}^{\pi} \sin ax \cos bx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(a+b)x + \sin(a-b)x dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin(a+b)x dx = 0 \\
\langle \cos ax, \cos bx\rangle &= \int_{-\pi}^{\pi} \cos ax \cos bx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(a+b)x + \cos(a-b)x dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos(a+b)x + 1 dx = \pi \\
\langle \sin ax, \sin bx\rangle &= \int_{-\pi}^{\pi} \sin ax \sin bx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(a-b)x - \cos(a+b)x dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos(a+b)x dx = \pi
\end{aligned}$$

Discrete case

- Basis function: $\{1, \cos(x), \ldots, \cos(\ell x), \sin(x), \ldots, \sin(\ell x)\}$.
- Consider $x_k = k(2\pi/n)$ for $k = 0, 1, \ldots, n-1$ in $[0, 2\pi)$.
  Each function is defined as an $n \times 1$ vector. For example,
  $$f_1(x) = 1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, f_2(x) = \cos(x) = \begin{pmatrix} \cos(x_1) \\ \cos(x_2) \\ \vdots \\ \cos(x_{n-1}) \end{pmatrix} = \begin{pmatrix} \cos(0\pi/n) \\ \cos(2\pi/n) \\ \vdots \\ \cos(\frac{2(n-1)\pi}{n}) \end{pmatrix}$$
- Those basis functions on x_k are orthogonal.

$$\langle 1, 1 \rangle = \sum_{j=0}^{n-1} 1 = n$$

$$
\begin{aligned}
\langle 1, \cos(mx) \rangle &= \sum_{j=0}^{n-1} \cos(mj\frac{2\pi}{n}) = 0 \\
\langle 1, \sin(mx) \rangle &= \sum_{j=0}^{n-1} \sin(mj\frac{2\pi}{n}) = 0 \\
\langle \cos(kx), \cos(mx) \rangle &= \frac{1}{2}\sum_{j=0}^{n-1} [\cos((k+m)j\frac{2\pi}{n}) + \cos((k-m)j\frac{2\pi}{n})] \\
&= \begin{cases} 0, & k-m \text{ and } k+m \text{ are not multiple of } n; \\ n/2, & k-m \text{ or } k+m \text{ are multiple of } n; \end{cases} \\
\langle \cos(kx), \sin(mx) \rangle &= \frac{1}{2}\sum_{j=0}^{n-1} [\sin((k+m)j\frac{2\pi}{n}) - \sin((k-m)j\frac{2\pi}{n})] = 0 \\
\langle \sin(kx), \sin(mx) \rangle &= \frac{1}{2}\sum_{j=0}^{n-1} [\cos((k-m)j\frac{2\pi}{n}) - \cos((k+m)j\frac{2\pi}{n})] \\
&= \begin{cases} 0, & k-m \text{ and } k+m \text{ are not multiple of } n; \\ n/2, & k-m \text{ or } k+m \text{ are multiple of } n; \end{cases}
\end{aligned}
$$

Fourier Interpolation and Approximation

Fourier interpolation

- Find $a_0, a_1, a_2 \cdots, a_m, b_1, b_2, \cdots, b_m$, such that

  $f(x) = \frac{a_0}{2} + a_1 \cos(x) \cdots + a_m \cos(mx) + b_1 \sin(x) \cdots + b_m \sin(mx)$

  interpolates $(0, x_0), (\frac{2\pi}{n}, x_1), \cdots ((n-1)\frac{2\pi}{n}, x_{n-1})$, where $n = 2m + 1$.

- Since the trigonometric functions are orthogonal at given points,

$$a_j = \frac{2}{n} \sum_{k=0}^{n-1} x_k \cos(jk\frac{2\pi}{n}), b_j = \frac{2}{n} \sum_{k=0}^{n-1} x_k \sin(jk\frac{2\pi}{n})$$

  for $j = 0, \ldots, m$.

Fourier approximation

- Find $a_0, a_1, a_2 \cdots, a_m, b_1, b_2, \cdots, b_m$, such that

  $f(x) = \frac{a_0}{2} + a_1 \cos(x) \cdots + a_m \cos(mx) + b_1 \sin(x) \cdots + b_m \sin(mx)$

  approximates $(0, x_0), (\frac{2\pi}{n}, x_1), \cdots ((n-1)\frac{2\pi}{n}, x_{n-1})$, where $n > 2m + 1$.

- Since the trigonometric functions are orthogonal at given points,

$$a_j = \frac{2}{n} \sum_{k=0}^{n-1} x_k \cos(jk\frac{2\pi}{n}), b_j = \frac{2}{n} \sum_{k=0}^{n-1} x_k \sin(jk\frac{2\pi}{n})$$

  for $j = 0, \ldots, m$.

Complex representation

- For $j = 0, \ldots, m$, the computation of

$$a_j = \frac{2}{n}\sum_{k=0}^{n-1} x_k \cos(jk\frac{2\pi}{n}), b_j = \frac{2}{n}\sum_{k=0}^{n-1} x_k \sin(jk\frac{2\pi}{n})$$

can be simplified by using Euler's formula.

$$c_j = \frac{a_j}{2} + i\frac{b_j}{2} = \frac{1}{n}\sum_{k=0}^{n-1} x_k(\cos(jk\frac{2\pi}{n}) + i\sin(jk\frac{2\pi}{n})) = \frac{1}{n}\sum_{k=0}^{n-1} x_k e^{ijk2\pi/n}$$

- Let $w = e^{i2\pi/n}$.

$$c_j = \frac{1}{n}\sum_{k=0}^{n-1} x_k w^{jk}$$

Complex vectors and matrices

- The inverse of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ is $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$.
  - $\bar{z} = a - ib$ is called the conjugate of $z = a + ib$.
  - $|z| = |\bar{z}| = |a + ib| = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2 + b^2}$.

- The conjugate transpose of a complex vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is

  $\mathbf{x}^H = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{pmatrix}$

- The inner product of two complex vectors is defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \mathbf{y}$

- The conjugate transpose of a complex matrix $\mathbf{W} = \{w_{i,j}\}$ is $\mathbf{W}^H = \{\bar{w}_{j,i}\}$
- An $n \times n$ complex matrix $\mathbf{U}$ is *unitary* if $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$. (compare to the orthogonal matrix of real matrices.)
- An $n \times n$ complex matrix $\mathbf{U}$ is *Hermitian* if $\mathbf{U}^H = \mathbf{U}$. (compare to the symmetric matrix of real matrices.)

Discrete Fourier transform

- For a vector $\mathbf{x}$ with n elements, $x_0, x_1, \cdots, x_{n-1}$, the Fourier transform produces an n elements vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \end{pmatrix}, X_k = \sum_{j=0}^{n-1} e^{-ij\frac{k2\pi}{n}} x_j$$

  - Basis functions are $e^0, e^{-ix}, e^{-i2x}, \cdots, e^{-i(n-1)x}$
  - Equidistant points in $[0, 2\pi)$ are $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, (n-1)\frac{2\pi}{n}$.

- The inverse discrete Fourier transform is $x_k = \frac{1}{n} \sum_{j=0}^{n-1} e^{ij\frac{k2\pi}{n}} X_j$

- Example, $n = 4$, Fourier transform is $\mathbf{X} = \mathbf{W}_4\mathbf{x}$, where

$$\mathbf{W}_4 = \begin{pmatrix} e^{-0\frac{0\pi}{4}} & e^{-i\frac{0\pi}{4}} & e^{-2i\frac{0\pi}{4}} & e^{-3i\frac{0\pi}{4}} \\ e^{-0\frac{2\pi}{4}} & e^{-i\frac{2\pi}{4}} & e^{-2i\frac{2\pi}{4}} & e^{-3i\frac{2\pi}{4}} \\ e^{-0\frac{4\pi}{4}} & e^{-i\frac{4\pi}{4}} & e^{-2i\frac{4\pi}{4}} & e^{-3i\frac{4\pi}{4}} \\ e^{-0\frac{6\pi}{4}} & e^{-i\frac{6\pi}{4}} & e^{-2i\frac{6\pi}{4}} & e^{-3i\frac{6\pi}{4}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

  - $\mathbf{W}_4$ is symmetric, but not Hermitian. Neither unitary.

- Inverse Fourier transform is $\mathbf{x} = \frac{1}{4}\mathbf{W}_4^H\mathbf{X}(= \frac{1}{4}\mathbf{W}_4^H\mathbf{W}_4\mathbf{x} = \frac{4\mathbf{I}}{4}\mathbf{x})$

$$\mathbf{W}_4^H = \begin{pmatrix} e^{0\frac{0\pi}{4}} & e^{i\frac{0\pi}{4}} & e^{2i\frac{0\pi}{4}} & e^{3i\frac{0\pi}{4}} \\ e^{0\frac{2\pi}{4}} & e^{i\frac{2\pi}{4}} & e^{2i\frac{2\pi}{4}} & e^{3i\frac{2\pi}{4}} \\ e^{0\frac{4\pi}{4}} & e^{i\frac{4\pi}{4}} & e^{2i\frac{4\pi}{4}} & e^{3i\frac{4\pi}{4}} \\ e^{0\frac{6\pi}{4}} & e^{i\frac{6\pi}{4}} & e^{2i\frac{6\pi}{4}} & e^{3i\frac{6\pi}{4}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Fast Fourier Transform

Matrix form

$$c_j = \sum_{k=0}^{n-1} w^{jk} x_k$$

Let $w = e^{-i2\pi/8}$. Above equation can be rewritten as $\mathbf{W}_8\mathbf{x} = \mathbf{c}$.

$$\begin{pmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ w^0 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ w^0 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix}$$

FFT reduces the $O(n^2)$ computation into $O(n \log n)$.

Divide and Conquer

- Consider a smaller matrix $\mathbf{W}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^1 & w_4^2 & w_4^3 \\ 1 & w_4^2 & w_4^4 & w_4^6 \\ 1 & w_4^3 & w_4^6 & w_4^9 \end{pmatrix}$

  where $w_4 = e^{-i2\pi/4}$.

- Relation to w: $w^2 = (e^{-i2\pi/8})^2 = e^{-i2\pi/4} = w_4$.

$$\mathbf{W}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^4 & w^8 & w^{12} \\ 1 & w^6 & w^{12} & w^{18} \end{pmatrix}$$

Relation of $\mathbf{W}_8$ and $\mathbf{W}_4$

- Put all odd columns first, then all even columns
- Use permutation matrix $\mathbf{P}$

$$\mathbf{W}_8\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w^2 & w^4 & w^6 & w^1 & w^3 & w^5 & w^7 \\ 1 & w^4 & w^8 & w^{12} & w^2 & w^6 & w^{10} & w^{14} \\ 1 & w^6 & w^{12} & w^{18} & w^3 & w^9 & w^{15} & w^{21} \\ 1 & w^8 & w^{16} & w^{24} & w^4 & w^{12} & w^{20} & w^{28} \\ 1 & w^{10} & w^{20} & w^{30} & w^5 & w^{15} & w^{25} & w^{35} \\ 1 & w^{12} & w^{24} & w^{36} & w^6 & w^{18} & w^{30} & w^{42} \\ 1 & w^{14} & w^{30} & w^{42} & w^7 & w^{21} & w^{35} & w^{49} \end{pmatrix} = \begin{pmatrix} \mathbf{W}_{8(1,1)} & \mathbf{W}_{8(1,2)} \\ \mathbf{W}_{8(2,1)} & \mathbf{W}_{8(2,2)} \end{pmatrix}$$

- $\mathbf{W}_{8(1,1)} = \mathbf{W}_4$

- Observe that
  - $w^0 = 1 = w^8 = w^{16} = w^{24} = w^{32} = w^{40} = w^{48}$,
  - and $w^4 = -1, w^5 = -w, w^6 = -w^2, w^7 = -w^3$.

$$\mathbf{W}_8\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w^2 & w^4 & w^6 & w^1 & w^3 & w^5 & w^7 \\ 1 & w^4 & w^8 & w^{12} & w^2 & w^6 & w^{10} & w^{14} \\ 1 & w^6 & w^{12} & w^{18} & w^3 & w^9 & w^{15} & w^{21} \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & w^2 & w^4 & w^6 & -w^1 & -w^3 & -w^5 & -w^7 \\ 1 & w^4 & w^8 & w^{12} & -w^2 & -w^6 & -w^{10} & -w^{14} \\ 1 & w^6 & w^{12} & w^{18} & -w^3 & -w^9 & -w^{15} & -w^{21} \end{pmatrix}$$

- $\mathbf{W}_{8(2,1)} = \mathbf{W}_{8(1,1)} = \mathbf{W}_4$, and $\mathbf{W}_{8(2,2)} = -\mathbf{W}_{8(1,2)}$

- If define $\mathbf{D}_4 = \begin{pmatrix} 1 & & & \\ & w^1 & & \\ & & w^2 & \\ & & & w^3 \end{pmatrix}$

$$\begin{aligned} \mathbf{D}_4\mathbf{W}_4 &= \begin{pmatrix} 1 & & & \\ & w^1 & & \\ & & w^2 & \\ & & & w^3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^4 & w^8 & w^{12} \\ 1 & w^6 & w^{12} & w^{18} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ w^1 & w^3 & w^5 & w^7 \\ w^2 & w^6 & w^{10} & w^{14} \\ w^3 & w^9 & w^{15} & w^{21} \end{pmatrix} = \mathbf{W}_{8(1,2)} = -\mathbf{W}_{8(2,2)} \end{aligned}$$

- $\mathbf{W}_8\mathbf{P} = \begin{pmatrix} \mathbf{W}_4 & \mathbf{D}_4\mathbf{W}_4 \\ \mathbf{W}_4 & -\mathbf{D}_4\mathbf{W}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{D}_4 \\ \mathbf{I}_4 & -\mathbf{D}_4 \end{pmatrix} \begin{pmatrix} \mathbf{W}_4 & \\ & \mathbf{W}_4 \end{pmatrix}$

Recursive algorithm

- Original problem $\mathbf{W}_8\mathbf{x} = \mathbf{c}$ can be expressed as

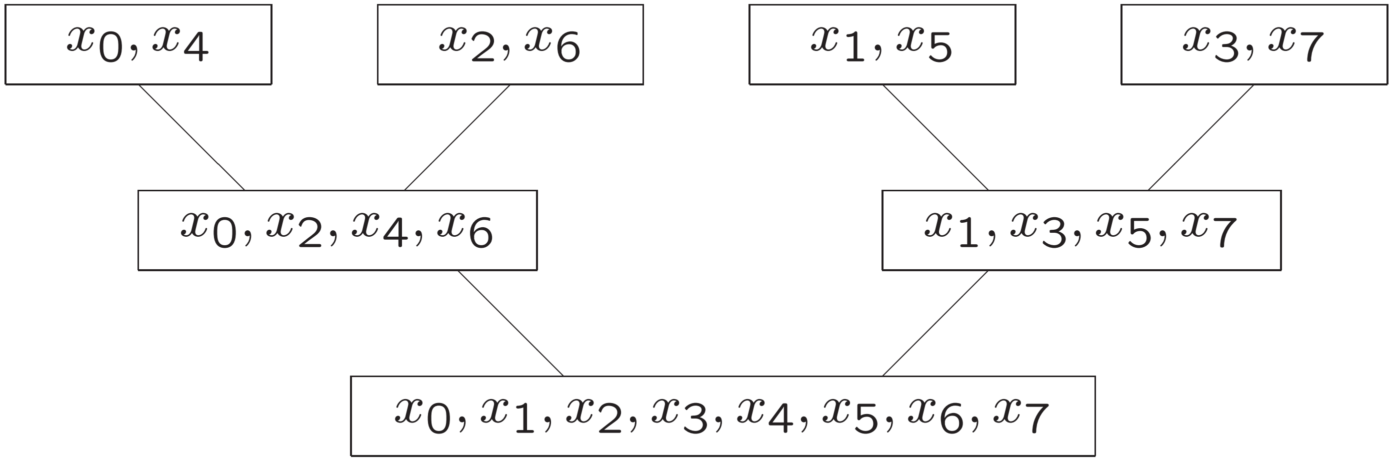
$$\mathbf{W}_8\mathbf{x} = \mathbf{W}_8\mathbf{P}\mathbf{P}^{-1}\mathbf{x} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{D}_4 \\ \mathbf{I}_4 & -\mathbf{D}_4 \end{pmatrix} \begin{pmatrix} \mathbf{W}_4 & \\ & \mathbf{W}_4 \end{pmatrix} \mathbf{P}^{-1}\mathbf{x} = \mathbf{c}$$

- $\mathbf{P}$ is an orthogonal matrix, whose inverse is $\mathbf{P}^T$.
  - Matlab syntax: $\mathbf{P}^T\mathbf{x} = [\mathbf{x}(1:2:\text{end}); \mathbf{x}(2:2:\text{end})]$

- FFT Algorithm for computing $\mathbf{W}_8\mathbf{x} = \mathbf{c}$

1. Compute $\begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{W}_4\mathbf{x}(1:2:\text{end}) \\ \mathbf{W}_4\mathbf{x}(2:2:\text{end}) \end{pmatrix}$ recursively.
2. $\mathbf{c} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{D}_4 \\ \mathbf{I}_4 & -\mathbf{D}_4 \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 + \mathbf{D}_4\mathbf{c}_2 \\ \mathbf{c}_1 - \mathbf{D}_4\mathbf{c}_2 \end{pmatrix}$

- Recursive order:



- Time complexity: $O(n \log n)$
  - Suppose the time of the algorithm is $T(n)$
  - Step 1 requires $2T(n/2)$, step 2 requires $O(n)$.

$$T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = n \log n$$

Non-recursive algorithm

- Arrange the elements as $a_0, a_4, a_2, a_6, a_1, a_5, a_3, a_7$.

element	binary index	bit reverse	reversed order
x_0	000	000	x_0
x_1	001	100	x_4
x_2	010	010	x_2
x_3	011	110	x_6
x_4	100	001	x_1
x_5	101	101	x_5
x_6	110	011	x_3
x_7	111	111	x_7

- Algorithm:

1. $\mathbf{c}$ =BitReverse($\mathbf{x}$)
2. For $s = 0 : (\lg n) - 1$
3. $\quad m = 2^s$
4. $\quad w = e^{-i\pi/m}$, $\mathbf{D}_m = \begin{pmatrix} 1 & & & \\ & w & & \\ & & \ddots & \\ & & & w^{m-1} \end{pmatrix}$
5. $\quad$ For $k = 0 : 2m : n - 1$
6. $\quad\quad \mathbf{c}_1 = \mathbf{c}(k : k + m - 1)$
7. $\quad\quad \mathbf{c}_2 = \mathbf{D}_m \mathbf{c}(k + m : k + 2m - 1)$
8. $\quad\quad \mathbf{c}(k : k + 2m) = \begin{pmatrix} \mathbf{c}_1 + \mathbf{c}_2 \\ \mathbf{c}_1 - \mathbf{c}_2 \end{pmatrix}$