1 Condition number estimation algorithm

The condition number estimation algorithm in textbook 126 is based on the 1-norm estimation algorithm of $\|\mathbf{A}^{-1}\|$. Once it has $\|\mathbf{A}^{-1}\|_1$, the condition number equals to $\|\mathbf{A}\|_1\|\mathbf{A}^{-1}\|_1$.

Here we only discuss the algorithm of estimating $\|\mathbf{A}\|_1$. By replacing all the matrix-vector multiplication $\mathbf{Ax}, \mathbf{A}^T\mathbf{x}$ by linear system solving $\mathbf{Ay} = \mathbf{x}, \mathbf{A}^T\mathbf{y} = \mathbf{x}$, the algorithm can estimate $\|\mathbf{A}^{-1}\|_1$.

The algorithm of estimating $\|\mathbf{A}\|_1$ is described as follows

1. $\mathbf{v}$ is an initial vector with $\|\mathbf{v}\|_1 = 1$
2. For $k = 1, 2, \ldots$
   (a) $\mathbf{u} = \mathbf{Av}$.
   (b) $\mathbf{w} = \text{sign}(\mathbf{u})$
   (c) $\mathbf{x} = \mathbf{A}^T\mathbf{w}$
   (d) Compute $\|\mathbf{x}\|_\infty$ and let j be the index of $|x_j| = \|\mathbf{x}\|_\infty$.
   (e) if ($\|\mathbf{x}\|_\infty \leq \|\mathbf{u}\|_1$) return $\|\mathbf{u}\|_1$.
   (f) $\mathbf{v} = \mathbf{e}_j$, and loop-back

Initially, we pick an arbitrary vector $\mathbf{v}$ with $\|\mathbf{v}\|_1 = 1$. After that, we will search $j_1, j_2 \cdots$, such that

$$\|\mathbf{Ae}_{j_1}\|_1 > \|\mathbf{Ae}_{j_2}\|_1 > \cdots$$

where $\mathbf{e}_{j_i}$ is a vector with the j_ith element 1 and others 0. The reason is the definition of matrix 1-norm is

$$\|\mathbf{A}\|_1 = \max_{j\in\{1,2,\ldots,n\}} \|\mathbf{Ae}_j\|_1.$$

Therefore, in each loop, what the algorithm does is to find a j such that

$$\|\mathbf{Ae}_j\|_1 > \|\mathbf{Av}\|_1 = \|\mathbf{u}\|_1 \tag{1}$$

until no such j can be found.

In step 2.(b), the algorithm defines an auxiliary vector $\mathbf{w} = \text{sign}(\mathbf{u})$ ($w_i = \text{sign}(u_i)$). By using that, $\|\mathbf{u}\|_1$ can be expressed as $\mathbf{w}^T\mathbf{u}$. Since elements in

$\mathbf{w}$ are ± 1 (or 0), it is always true that $\|\mathbf{A}\mathbf{e}_j\|_1 \geq |\mathbf{w}^T\mathbf{A}\mathbf{e}_j|$ for any j. As the result, the problem can be simplified as: Find a j such that

$$|\mathbf{w}^T\mathbf{A}\mathbf{e}_j| > \|\mathbf{u}\|_1. \tag{2}$$

The problem in equation (2) is much simpler than the one in (1). In (1), one needs to go through entire matrix $\mathbf{A}$, but in (2), one only needs to search a vector $\mathbf{w}^T\mathbf{A}$. Step 2.(c) defines $\mathbf{x} = \mathbf{A}^T\mathbf{w}$. Equation (2) is equivalent to find an element x_j in $\mathbf{x}$ such that $|x_j| > \|\mathbf{u}\|_1$. An best choice is the maximum absolute valued element in $\mathbf{x}$, which equals to $\|\mathbf{x}\|_\infty$.

Step 2.(e) compares $\|\mathbf{x}\|_\infty$ and $\|\mathbf{u}\|_1$. If $\|\mathbf{x}\|_\infty$ is larger, then one finds the j, since

$$\|\mathbf{A}\mathbf{e}_j\|_1 \geq |\mathbf{w}^T\mathbf{A}\mathbf{e}_j| = \|\mathbf{x}\|_\infty > \|\mathbf{u}\|_1.$$

Otherwise, the program returns $\|\mathbf{u}\|_1$ as the estimation.