Solutions for HW4

(1) By applying a *modified* Gram-Schmidt orthogonalizarion process on the matrix A given below.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$$

We have (there may have different versions of Modified Gram-Schmidt)

(i) $\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

(ii) $\mathbf{z}_2 = \mathbf{a}_2 - (\mathbf{q}_1^t \mathbf{a}_2)\mathbf{q}_1$, $\mathbf{q}_2 = \frac{\mathbf{z}_2}{\|\mathbf{z}_2\|_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

(iii) $\mathbf{y}_3 = \mathbf{a}_3 - \langle \mathbf{a}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$; $\mathbf{z}_3 = \mathbf{y}_3 - \langle \mathbf{a}_3, \mathbf{q}_2 \rangle \mathbf{q}_2 =$ $\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, so $\mathbf{q}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then, we have

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 4 \end{bmatrix}$$

(2)

$$B = \begin{bmatrix} -1 & 0 & 3 \\ -3 & 2 & 3 \\ -6 & 0 & 8 \end{bmatrix}, \quad det(\lambda I - B) = \begin{bmatrix} \lambda+1 & 0 & -3 \\ 3 & \lambda-2 & -3 \\ 6 & 0 & \lambda-8 \end{bmatrix}$$

Let the determinant $det(\lambda I - B) = 0$, we want to solve $det(\lambda I - B) = (\lambda+1)(\lambda-2)(\lambda-8) + 18(\lambda-2) = (\lambda-2)^2(\lambda-5) = 0$. Thus, $\lambda = 2, 2, 5$.

For $\lambda = 2$, we have corresponding eigenvectors $\{[a, b, a]^T \mid any\ constant\ a\ and\ b,\ a^2 + b^2 \neq 0\}$. Thus, we can pick up $a = \frac{1}{\sqrt{2}}$ and $b = 0$ or $a = 0$, $b = 1$ as two eigenvectors corresponding to $\lambda = 2$.

For $\lambda = 5$, we have a corresponding eigenvector $\{a[1, 1, 2]^T \mid any\ constant\ \ a \neq 0\}$, let $a = \frac{1}{\sqrt{5}}$. Denote

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \quad and\ \ U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then,

$$U^{-1}BU = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(3) For real $m \times m$ matrices A and U, $\kappa_2(U) = \|U\|_2 \|U^{-1}\|_2$, $\|U\|_2 = max_{\|\mathbf{x}\|_2=1}\{\|U\mathbf{x}\|_2\}$.

(i) If U is orthogonal, then $U^T U = I$, so $U^{-1} = U^T$, then

$$\begin{aligned} \|U\|_2 &= max_{\|\mathbf{x}\|_2=1}\{\|U\mathbf{x}\|_2\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\|U\mathbf{x}\|_2^2\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^T U^T U \mathbf{x}\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^T (U^T U = I)\mathbf{x}\} \\ &= 1 \end{aligned}$$

Similarly,

$$\begin{aligned}\|U^{-1}\|_2 &= max_{\|\mathbf{x}\|_2=1}\{\|U^{-1}\mathbf{x}\|_2\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\|U^{-1}\mathbf{x}\|_2^2\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^T(U^{-1})^T U^{-1}\mathbf{x}\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^T(UU^{-1} = I)\mathbf{x}\} \\ &= 1\end{aligned}$$

Thus, $\kappa_2(U) = \|U\|_2\|U^{-1}\|_2 = 1 \times 1 = 1$. Moveover, for any m-dimensional column vector $\mathbf{x}$ with $\|\mathbf{x}\|_2 = 1$, we can have an m-dimensional column vector $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1$ such that $\mathbf{y} = U\mathbf{x}$, and vice versa. Thus,

$$\begin{aligned}\|AU\|_2 &= max_{\|\mathbf{x}\|_2=1}\{\|AU\mathbf{x}\|_2\} \\ &= max_{\|\mathbf{y}\|_2=1}\{\|A\mathbf{y}\|_2\} \\ &= \|A\|_2\end{aligned}$$

$$\begin{aligned}\|UA\|_2 &= max_{\|\mathbf{x}\|_2=1}\{\|UA\mathbf{x}\|_2\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\|UA\mathbf{x}\|_2^2\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^T A^T(U^T U = I)A\mathbf{x}\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^T A^T A\mathbf{x}\} \\ &= max_{\|\mathbf{x}\|_2=1}\{\|A\mathbf{x}\|_2\} \\ &= \|A\|_2\end{aligned}$$

Similar arguments, we can prove that $\|(AU)^{-1}\|_2 = \|A^{-1}\|_2$, and $\|(UA)^{-1}\|_2 = \|A^{-1}\|_2$. Therefore, $\kappa_2(AU) = \|AU\|_2\|(AU)^{-1}\|_2 = \|A\|_2\|A^{-1}\|_2 = \kappa(A)$.

(ii) By definition, $\|A\|_2 = max_{\|\mathbf{x}\|_2\neq 0}\left\{\frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}\right\}$, then for any $\mathbf{x} \neq \mathbf{0}$, we have $\|A\mathbf{x}\|_2 \leq \|A\|_2\|\mathbf{x}\|_2$, we know $\mathbf{x} = (A^{-1}A)\mathbf{x}$, then $\|\mathbf{x}\|_2 = \|(A^{-1}A)\mathbf{x}\|_2 = \|A^{-1}[A\mathbf{x}]\|_2 \leq \|A^{-1}\|_2\|A\mathbf{x}\|_2 \leq \|A^{-1}\|_2\|A\|_2\|\mathbf{x}\|_2$, thus $1 \leq \|A^{-1}\|_2\|A\|_2$, so $\kappa_2(A) \geq 1$ for any $m \times m$ invertible matrix A.

(4)

$$Let A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then the least squares solution can be computed by

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} = (A^T A)^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \end{bmatrix}$$

(5) I'll do this when I have time.