I. Let \( X \) be a discrete random variable (r.v.) having the probability mass function (pmf) \( f(x) \), then the mean \( \mu \), variance \( \sigma^2 \), and the corresponding moment-generating function \( \phi(t) \) are defined as follows.

\[
\mu = E[X] = \sum_x x f(x)
\]

\[
\sigma^2 = Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2
\]

\[
\phi(t) = E[e^{tX}] = \sum_x e^{tx} f(x)
\]

Moreover, we know that

\[
\mu = \phi'(0), \quad \text{and} \quad \sigma^2 = \phi''(0) - [\phi'(0)]^2
\]

For a discrete type of r.v. \( X \) which has one of the following probability mass functions, derive the formula for the moment-generating function, and compute the mean and variance, respectively.

**Binomial** \( f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n \)

**Geometric** \( f(x) = (1-p)^{x-1} p, \quad x = 1, 2, \ldots \)

**Negative Binomial** \( f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots \)

**Poisson** \( f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \)

**Hypergeometric (optional)** \( f(x) = \binom{N}{x} \binom{M}{n-x} / \binom{N+M}{n} \) \( 0 \leq x \leq n, \quad x \leq N, \quad n-x \leq M \)

**Uniform** \( f(x) = \frac{1}{m}, \quad x = 1, 2, \ldots, m \)
II. Let $X$ be a continuous random variable (r.v.) having the probability density function (pdf) $f(x)$, then the mean $\mu$, variance $\sigma^2$, and the corresponding moment-generating function $\phi(t)$ are defined as follows.

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x) \, dx$$

$$\sigma^2 = Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

$$\phi(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx$$

Moreover, we know that

$$\mu = \phi'(0), \text{ and } \sigma^2 = \phi''(0) - [\phi'(0)]^2$$

For a continuous type of r.v. $X$ which has one of the following probability density functions, derive the formula for the moment-generating function, and compute the mean and variance, respectively.

**Uniform** $U(a, b)$  
$f(x) = \frac{1}{b-a}, a \leq x \leq b$

**Exponential**  
$f(x) = \frac{1}{\theta} e^{-x/\theta}, 0 < x < \infty$

**Gamma**  
$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha x^{\alpha-1}e^{-x/\theta}}, 0 < x < \infty$

**$\chi^2(r)$ Chi-Square**  
$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{(r/2)-1}e^{-x/2}, 0 < x < \infty$

**$N(\mu, \sigma^2)$ Normal**  
$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, -\infty < x < \infty$

**Beta** ($\times$)  
$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$

III.

(a) Let $X_r \sim \chi^2(r)$ have the $\chi^2$ distribution with $r$ degrees of freedom and let $Y_j \sim N(0, \sigma_j^2)$. Write Matlab codes to plot the $\chi^2(r) - distributions$ with $r = 4, 7, 8$ in the same frame.

(b) Let $Y_j \sim N(\mu, \sigma_j^2)$, plot the normal distributions with mean $\mu = 0$ and $\sigma_1^2 = 1, \sigma_2^2 = 4, \sigma_3^2 = 16, \sigma_4^2 = 100$ in the same frame.

(c) Turn in your Matlab codes and a 1-page output for (a) and (b).