Solutions for Test 2 of ISA5305, Fall 2019

Due in class of Thursday, December 26, 2019

Name : ________________ *ID* : ________________

(15%) 1(a) Let X be a continuous type of random variable that takes only nonnegative values. For any $\beta > 0$, prove that

$$P(X \geq \beta) \leq \frac{E(X)}{\beta}$$

$$\begin{aligned} E(X) &= \int_0^\infty xf(x)dx &= \int_0^\beta xf(x)dx + \int_\beta^\infty xf(x)dx \\ &\geq \int_\beta^\infty xf(x)dx &\geq \beta \int_\beta^\infty f(x)dx \\ &= \beta P(X \geq \beta) \end{aligned}$$

Therefore, $P(X \geq \beta) \leq \frac{E(X)}{\beta}$

(15%) 1(b) Let X be a continuous type of random variable with mean $E(X) = \mu$ and variance $Var(X) = \sigma^2$. For any $\tau > 0$, show that

$$P(|X - \mu| \geq \tau) \leq \frac{\sigma^2}{\tau^2}$$

$$\begin{aligned} Var(X) &= \int_{-\infty}^\infty (x-\mu)^2 f(x)dx \\ &= \int_{|x-\mu|\geq\tau} (x-\mu)^2 f(x)dx + \int_{|x-\mu|<\tau} (x-\mu)^2 f(x)dx \\ &\geq \int_{|x-\mu|\geq\tau} \tau^2 f(x)dx \\ &= \tau^2 \int_{|x-\mu|\geq\tau} f(x)dx \\ &= \tau^2 P(| X - \mu | \geq \tau) \end{aligned}$$

Therefore, $P(| X - \mu | \geq \tau) \leq \frac{Var(X)}{\tau^2} \leq \frac{\sigma^2}{\tau^2}$

(30%) 2. Fill the following blanks (no partial credits).

(a) Let $\{X_i,\ 1 \le i \le 9\}$ be a random sample of size 9 from $N(2,1)$. Define $\overline{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $Y = \sum_{i=1}^{9}(X_i - 2)^2$. Then

the moment-generating function $M_{\overline{X}}(t) =$ $\underline{e^{2t+(t^2/18)}}$

the moment-generating function $M_Y(t) =$ $\underline{\frac{1}{(1-2t)^{9/2}}}$

(b) Let $\{Z_i \sim N(0,1),\ 1 \le i \le 10\}$ be a random sample of size 10. Define $V = \sum_{i=1}^{10} Z_i$ and $W = \sum_{i=1}^{10} Z_i^2$. Then

the probability density function of V, $f_V(x) =$ $\underline{\frac{1}{\sqrt{20\pi}} e^{-x^2/20}, \quad -\infty < x < \infty}$

the moment-generating function $M_W(t) =$ $\underline{\frac{1}{(1-2t)^5}}$ and the p.d.f. of W, $f_W(x) =$ $\underline{\frac{1}{\Gamma(5)2^5} x^4 e^{-x/2}}$

(c) Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size n from the *exponential distribution* with $E(X_n) = 3$. Define $Y = \sum_{i=1}^{n} X_i$. Then

the p.d.f. of Y, $f_Y(y) =$ $\underline{\frac{1}{\Gamma(n)3^n} y^{n-1} e^{-y/3}, \ y > 0}$

the moment-generating function of Y, $\phi_Y(t) =$ $\underline{\frac{1}{(1-3t)^n}}$

(d) Let $Z \sim N(0,1)$. Define $Y = 3Z + 2$, then

the p.d.f. of Y, $f_Y(y) =$ $\underline{\frac{1}{\sqrt{18\pi}} exp[-\frac{(y-2)^2}{18}], \quad -\infty < y < \infty}$

the moment-generating function of Y, $\phi_Y(t) =$ $\underline{e^{2t+\frac{9t^2}{2}}}$

(e) Let $\{X_i \sim \chi^2(1),\ 1 \le i \le n\}$ be a random sample and define $W_n = (\sum_{i=1}^{n} X_i - n)/\sqrt{2n}$. According to the Central Limit Theorem,

the limiting distribution function of W_n $limit_{n\to\infty} P(W_n \le w) =$ $\underline{\int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad = \quad \Phi(w)}$

the limiting moment-generating function is $limit_{n\to\infty} M_{W_n}(t) =$ $\underline{e^{t^2/2}}$

(10%) 3. Let the r.v. X have the p.d.f. $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, where $\alpha, \beta > 0$ are known positive integers.

Find $E[X]$ and $Var[X]$.

Hint: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ and $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Proof:

$$\begin{aligned} E[X] &= \int_0^1 x f(x) dx &&= \int_0^1 x \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1} dx &&= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\ &= \frac{\alpha}{\alpha+\beta)} \end{aligned}$$

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$$\begin{aligned} E[X^2] &= \int_0^1 x^2 f(x) dx &&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1} dx \\ &= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} &&= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \end{aligned}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(15%) 4. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample $\{X_1, X_2, \cdots, X_n\}$ from the uniform distribution U(0,1).

(a) Find the probability density function of $X_{(1)}$.

(b) Use the results of **(a)** to find $E[X_{(1)}]$.

(c) Find the probability density function of $X_{(n)}$.

(d) Use the result of **(c)** to find $E[X_{(n)}]$.

Hint: Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample from $U(0,1)$, then $X_{(1)} = min\{X_1, X_2, \cdots, X_n\}$ and $X_{(n)} = max\{X_1, X_2, \cdots, X_n\}$.

Proof:

$$\begin{aligned} P(X_{(1)} \leq x) &= 1 - P(X_{(1)} > x) \\ &= 1 - \Pi_{j=1}^{n} P(X_{(j)} > x) \ , \\ &= 1 - (1-x)^n \end{aligned}$$

then,

$$\begin{aligned} (a) \qquad f_{X_{(1)}}(x) &= n(1-x)^{n-1}, \quad 0 < x < 1 \\ (b) \qquad E(X_{(1)}) &= \textstyle\int_0^1 xn(1-x)^{n-1} dx \qquad . \\ = nB(2,n) &= \tfrac{1}{n+1} \end{aligned}$$

$$\begin{aligned} P(X_{(n)} \leq x) &= \Pi_{j=1}^{n} P(X_{(j)} \leq x) \\ &= \Pi_{j=1}^{n} x \qquad , \\ &= x^n \end{aligned}$$

then,

$$\begin{aligned} (c) \qquad f_{X_{(n)}}(x) &= nx^{n-1}, \quad 0 < x < 1 \\ (d) \qquad E(X_{(n)}) &= \textstyle\int_0^1 nx^n dx \qquad . \\ &= \tfrac{n}{n+1} \end{aligned}$$

(15%) 5. Let X have the p.d.f. $f(x) = \beta x^{\beta-1}, \quad 0 < x < 1$ for a given $\beta > 0$. Define $Y = -3\beta ln(X)$.

(a) Compute the distribution function of Y, $P(Y \leq y),\ for\ 0 < y < \infty$.

(b) Find the moment-generating function $M_Y(t)$.

(c) Find $E(Y)$ and $Var(Y)$.

Proof:

$$\begin{aligned} P(Y \leq y) &= P(-3\beta \ln(X) \leq y) \\ &= P(X \geq e^{-y/3\beta}) \\ &= \int_{e^{-y/3\beta}}^{1} \beta x^{\beta-1} dx \\ &= 1 - e^{-y/3} \end{aligned},$$

then, $(a)\quad f_Y(y) = \frac{1}{3}e^{-y/3} \quad 0 < y < \infty$, Y is an exponential distribution with the parameter $\theta = 3$. $(b)\quad M_Y(t) = \frac{1}{1-3t}$. $(c)\quad E(Y) = 3\ \ and\ \ Var(Y) = 9$.